Q1. Prove that if a set $\{x_1, x_2, ..., x_n\}$ is orthogonal in an inner product space X, then it is linearly independent.

Q2. Show that the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly convergent on $[0, \pi]$.

Q3. Consider the orthogonal set $S = \{\sin nx, n \in \mathbb{N}\} \subset \mathfrak{L}^2[-\pi, \pi]$

(a) The function $f(x) = x \in \mathcal{L}^2[-\pi, \pi]$ can be written as a linear combination of elements in S. Find the coefficients α_k of this linear combination.

(b) Use part a and the parseval equality to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Q4. Consider the sequence of functions

$$f_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ n\frac{1-x}{n-1}, & \frac{1}{n} < x \leq 1 \end{cases},$$

where $n \in \mathbb{N}$.

(a) Sketch the graph of the function $f_n(x)$ for $n = 1, 2$

(b) Show that $f_n \in \mathcal{L}^2(0, 1)$ for all $n \in \mathbb{N}$.

(c) Find the limit $f(x)$ of $f_n(x)$ as $n \to \infty$. Does $f \in \mathcal{L}^2[0, 1]$? Justify your answer.

(d) Does $f_n(x)$ converge to $f(x)$ uniformly? Justify your answer.

(e) Does $f_n(x)$ converge to $f(x)$ in $\mathcal{L}^2[0, 1]$? Justify your answer.