King Saud University, Department of Mathematics

Final Exam, 2024/25, Course: M-316

Marks: 40, Time: 3 hours

Question 1

1. Find α so that the set of functions $\{1, x-1,\ (x-1)^2+\ \alpha\}$ is orthogonal on [0,2], and obtain the orthonormal set.
2. Find the pointwise limit of the sequence

$$g_n(x) = \begin{cases} \sqrt{n}, & 0 \le x < \frac{1}{n} \\ 0, & \frac{1}{n} \le x \le 1 \end{cases}.$$

What about limit of $g_n(x)$ in the space $L^2[0,1]$?

Question 2 Consider the boundary value problem

$$x^2u'' + \lambda u = 0, x \in [1, e],\ u(1) = u(e) = 0.$$

Write the equation in the above problem in Sturm–Liouville form, then find its eigenvalues and eigenfunctions on $[1, e]$ subject to the boundary conditions. Show that the eigenfunctions are orthogonal in the space $L^2_{\frac{1}{x^2}}[1, e]$

Question 3 Consider the piecewise defined $f(x)$:

$$f(x) = \begin{cases} x + \frac{\pi}{2}, & -\pi < x \le 0 \\ -x + \frac{\pi}{2}, & 0 \le x < \pi \end{cases}, \quad f(x + 2\pi) = f(x), x \in \mathbb{R}.$$

(a) Sketch the function f on the interval $[-3\pi, 3\pi]$.

(b) Find the Fourier series representation of f.

(c) Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Question 4 Consider the function $f(x) = e^{-a|x|}, x \in \mathbb{R}$

a) Show that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

b) Find the Fourier integral of f.

c) Using part b) to find $\int_0^{\infty} \frac{\cos(\xi x)}{a^2+\xi^2}\, d\xi$.

Question 5

a) Prove that the set $\{P_n : n \in \mathbb{N}\}$ is orthogonal in the space $L^2([-1,1])$ and find $\|P_n\|$, where.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}((x^2-1)^n), n = 0,1,2, \ldots$$

b) Expand the function x^2 in terms of Legendre polynomials on the interval $[-1,1]$.

c) Consider the Bessel function of the first kind of order ν

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m}, x > 0.$$

Show that

$$\frac{d}{dx}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x), \quad x > 0, \nu \ge 0.$$