King Saud University: Mathematics Department MATH-254
First Semester 1447 H Final Examination
Maximum Marks = 40 Time: 180 mins.

Name of the Student:————————— **I.D. No.** —————

Name of the Teacher:————————— **Section No.** —————

Note: Check the total number of pages are Six (6).

(15 Multiple choice questions and Two (2) Full questions)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one $(2 \times 15 = 30)$

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d															

Question No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

Question 1: The number of iterations required to achieve an approximation with accuracy 10^{-4} to the solution of $g(x) = 2^{-x}$ lying in the interval $[\frac{1}{3}, 1]$ and $x_0 = 1$ using the fixed-point method is:

(A) 16 **(B)** 18 **(C)** 19 **(D)** None of These

Question 2: Which of the following initial guess x_0, the Newton's method fails to approximate a root of the equation $x^2 - \ln x = x$:

(A) $x_0 = 1.7$ **(B)** $x_0 = 1.5$ **(C)** $x_0 = 1$ **(D)** None of These

Question 3: The first approximation for the root $\alpha = 0$ for the equation $e^{2x} - 2e^x + 1 = 0$ with $x_0 = 0.5$ using the first modified Newton's method is:

(A) 0.3032 **(B)** 0.1065 **(C)** 0.0165 **(D)** None of These

Question 4: The value of k for which the scheme $x_{n+1} = 3 - (2+k)x_n + kx_n^4$ will converge at least quadratically to the root $\alpha = 1$ is:

(A) $\frac{3}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{2}{5}$ **(D)** None of These

Question 5: If the Gaussian elimination method with partial pivoting is used to solve the linear system $0.03x_1 + 58.9x_2 = 59.2$, $5.31x_1 - 6.10x_2 = 47$, then $x_1 + x_2$ equals:

(A) 11 **(B)** 9 **(C)** 13 **(D)** None of These

Question 6: The first approximation for solving the linear system $A\mathbf{x} = [1, 3]^T$ using Gauss-Seidel iterative method with $A = \begin{pmatrix} -4 & 5 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{x}^{(0)} = [0.5, 0.5]^T$ is:

(A) $[0.3750, 1.3125]^T$ **(B)** $[0.3750, 1.2500]^T$ **(C)** $[1.3750, 1.2500]^T$ **(D)** None of These

Question 7: If $x^* = [0.5, 0.0]^T$ is an approximate solution for the linear system $2x_1 - x_2 = 1$, $x_1 + x_2 = 2$, then the relative error $\frac{||x - x^*||_\infty}{||x||_\infty} \leq k(A)\frac{||r||}{||b||}$ is bounded by:

(A) 1 **(B)** 1.5 **(C)** 2.25 **(D)** None of These

Question 8: Using the data points: $(0, f(0))$ and $(\pi, f(\pi))$. The error bound in the linear Lagrange interpolation polynomial of the function $f(x) = 2\cos x$ at $\frac{\pi}{2}$ is:

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi^2}{4}$ **(C)** $\frac{\pi^2}{2}$ **(D)** Nonef These

Question 9: Giving the following data

x	-2	-1	0	1
$f(x)$	-1	3	α	-1

If the first order divided difference $f[-1,0] = -3$, then the second order divided difference $f[-1,0,1]$ equals:

(A) 2 **(B)** -1 **(C)** 1 **(D)** None of These

Question 10: If $x_0 = 1, x_1 = 2, x_2 = 3$ and for a function $f(x)$, the divided differences are $f[x_1] = 2, f[x_2] = 3, f[x_0, x_1] = 1, f[x_1, x_2] = \frac{1}{2}, f[x_0, x_1, x_2] = -\frac{1}{6}$, then the approximation of $f(0.5)$ using the quadratic interpolation Newton formula is:

(A) 0.5730 **(B)** 0.7530 **(C)** 0.3750 **(D)** None of These

Question 11: Using the data points: $f(2.1) = 3.4321,\ f(2.3) = 4.3210,\ f(2.5) = 5.3210$ and $f(2.6) = 6.3215$. The approximate value of $f'(2.5)$ using three-point difference formula is:

(A) 5.2778 **(B)** 4.2778 **(C)** 5.8772 **(D)** None of These

Question 12: Using the data points: $f(2.1) = 3.4321,\ f(2.3) = 4.3210,\ f(2.5) = 5.3210$ and $f(2.7) = 7.3215$. The approximate value of $f''(2.3)$ using three-point difference formula is:

(A) 0.2778 **(B)** 2.7775 **(C)** 5.5550 **(D)** None of These

Question 13: If $f(0) = 3, f(1) = \frac{\alpha}{2}, f(2) = \alpha$, and Simpson's rule for $\int_0^2 f(x)\,dx$ gives 2, then the value of α is:

(A) 1.5 **(B)** 2.0 **(C)** 1.0 **(D)** None of These

Question 14: Given $y' = y - x^2 + 1,\ y(0) = 0.5$, the approximate value of $y(2)$ using Taylor's method of order 2 when $n = 2$ is:

(A) 2.7500 **(B)** 5.3476 **(C)** 5.8750 **(D)** None of These

Question 15: The exact solution of $4y' - y = 0,\ y(0) = 1$, is $y(x) = e^{(\frac{x}{4})}$, the absolute error in approximating the value of $y(0.5)$ using Runge-Kutta method of order 2 when $n = 1$ is:

(A) 0.0003 **(B)** 0.0300 **(C)** 0.0030 **(D)** None of These

Question 16: Construct the divided differences table for $f(x) = x^3 + 7x^2 + 1$ using the values $x = 1, 1.5, 3, 4, 5$. If the approximation of $f(3.5)$ by linear Newton's polynomial is 134, then use it to find the best approximation of $f(3.5)$ by using quadratic Newton's polynomial. Compute the absolute error and an error bound for your approximation.

Question 17: Find the approximation of $\int_0^{1.2} f(x)\,dx$ by using the following set of data points using the best composite integration rule:

x	0.0	0.1	0.22	0.3	0.4	0.5	0.6	0.72	0.8	0.9	1.0	1.1	1.2
$f(x)$	1.0	1.0950	1.1959	1.2553	1.3211	1.3776	1.4253	1.4718	1.4967	1.5216	1.5403	1.5536	1.5624

The function tabulated is $f(x) = x + \cos x$, compute an error bound and the absolute error for the approximation got by the best integration rule. How many subintervals approximate the given integral to within accuracy of 10^{-6} ?

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The Answer Tables for Q.1 to Q.15 : (MAth) Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

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a,b,c,d	B	A	C	C	B	B	A	C	B	D	B	A	A	B	B

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Question 16: Construct the divided differences table for $f(x) = x^3 + 7x^2 + 1$ using the values $x = 1, 1.5, 3, 4, 5$. If the approximation of $f(3.5)$ by linear Newton's polynomial is 134, then use it to find the best approximation of $f(3.5)$ by using quadratic Newton's polynomial. Compute the absolute error and an error bound for your approximation.

Solution. The divided differences of the given function are listed in Table 1.

Table 1: Divided differences table for given $f(x) = x^3 + 7x^2 + 1$.

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
0	1	9				
1	1.5	20.125	22.25			
2	3	91	47.25	12.5		
3	4	177	86	15.5	1	
4	5	301	124	19	1	0

Since the quadratic Newton's polynomial is

$$f(x) = p_2(x) = p_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2].$$

Given $x = 3.5$, so choosing the best points, $x_0 = 3, x_1 = 4, x_2 = 5$, and Table 1, we get

$$f(3.5) \approx p_2(3.5) = p_1(3.5) + (3.5 - 3)(3.5 - 4)f[3, 4, 5] = 134 - (0.25)(19) = 129.25,$$

which is the required approximation.
The absolute error is,

$$|f(3.5) - p_2(3.5)| = |129.625 - 129.25| = 0.3750.$$

Since the error bound for the quadratic polynomial $p_2(x)$ is

$$|f(x) - p_2(x)| \leq \frac{M}{3!}|(x - x_0)(x - x_1)(x - x_2)|,$$

where

$$M = \max_{3 \leq x \leq 5} |f^{(3)}(x)|.$$

Since the first three derivatives of the given function are

$$f'(x) = 3x^2 + 14x; \qquad f''(x) = 6x + 14; \qquad f^{(3)}(x) = 6,$$

so

$$M = \max_{3 \leq x \leq 5} |6| = 6.$$

Thus

$$|f(3.6) - p_2(3.6)| \leq \frac{M}{3!}|(3.5 - 3)(3.5 - 4)(3.5 - 5)| \leq \frac{(6)(0.3750)}{6} = 0.3750,$$

which is the required error bound for the approximation $p_2(3.5)$. •

Question 17: Find the approximation of $\int_0^{1.2} f(x)\ dx$ by using the following set of data points using the best composite integration rule:

x	0.0	0.1	0.22	0.3	0.4	0.5	0.6	0.72	0.8	0.9	1.0	1.1	1.2
$f(x)$	1.0	1.0950	1.1959	1.2553	1.3211	1.3776	1.4253	1.4718	1.4967	1.5216	1.5403	1.5536	1.5624

The function tabulated is $f(x) = x + \cos x$, compute an error bound and the absolute error for the approximation got by the best integration rule. How many subintervals approximate the given integral to within accuracy of 10^{-6} ?

Solution. To select the equally spaced set of data points, we have

x	0.0	0.3	0.6	0.9	1.2
$f(x)$	1.00	1.2553	1.4253	1.5216	1.5624

which gives, $h = 0.3$. So the best integration rule is composite Simpson's rule, which is for the five points ($n = 4$) can be written as

$$\int_0^{1.2} f(x)\ dx \approx S_4(f) = \frac{h}{3}\Big[f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\Big],$$

$$\int_0^{1.2} f(x)\ dx \approx 0.1\Big[1.0 + 4(1.2553 + 1.5216) + 2(1.4253) + 1.5624\Big] = 1.6521.$$

The first four derivatives of the function $f(x) = x + \cos x$ can be obtain as

$$f'(x) = 1 - \sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x.$$

Since $\eta(x)$ is unknown point in $(0, 1.2)$, therefore, the bound $|f^{(4)}|$ on $[0, 1.2]$ is

$$M = \max_{0\le x\le 1.2} |f^{(4)}| = \max_{0\le x\le 1.2} |\cos x| = 1.0,$$

at $x = 0$. Thus the error formula $|E_{S_n}(f)| \le \dfrac{h^4|b-a)}{180}M$, becomes

$$|E_{S_4}(f)| \le \frac{(0.3)^4(1.2)}{180}(1.0) = 0.000054.$$

We can easily computed the exact value of the given integral as

$$I(f) = \int_0^{1.2} (x + \cos x)\ dx = (\frac{x^2}{2} + \sin x)\Big|_0^{1.2} = 1.6521.$$

Thus the absolute error $|E|$ in our approximation is given as

$$|E| = |I(f) - S_4(f)| = |1.6521 - 1.6521| = 0.0000, \qquad 4dp.$$

To find the minimum subintervals for the given accuracy, we use the error formula such that

$$|E_{S_n}(f)| \le \frac{(b-a)^5}{180n^4}M \le 10^{-6},$$

where $h = (b-a)/n$. Since, $M = \max_{0\le x\le 1.2} |f^{(4)}(x)| = \max_{0\le x\le 1.2} |\cos(x)| = 1$, then

$$n^4 \ge \frac{((1.2)^5)(1)(10^6)}{180}, \qquad \text{or} \qquad n \ge \left(\frac{((1.2)^5)(1)(10^6)}{180}\right)^{1/4},$$

then solving for n, we obtain, $n \ge 10.8432$. Hence to get the required accuracy, we need 12 (even) subintervals. •