KING SAUD UNIVERSITY,DEPARTMENT OF MATHEMATICS

MATH 204. TIME: 3H, FULL MARKS: 40, FINAL EXAM

**Question 1.** [5,4] **a)** The population of a certain community is known to increase at a rate proportional to the number of people present at any time. If the population has doubled in 6 years, how long will it take to triple from its initial population.

**b)** Solve the initial value problem

$$x(1+y^2)dx + (1+x^2)^2 dy = 0, \quad y(-1) = 0.$$

**Question 2.** [4,5] **a)** Obtain the general solution of the differential equation

$$xy'y^{-3} + 2y^{-2} = x^3, \quad x > 0.$$

**b)** Use power series method to solve the initial value problem

$$(1-x^2)y'' - xy' + 9y = 0, \quad y(0) = 0, \ y'(0) = 3.$$

**Question 3.** [4,4,4] **a)** Find the largest interval for which the following initial value problem admits a unique solution

$$\begin{cases} \ln(x+3)y'' + \ln(1-4x^2)y' + y = 2x \\ y(\frac{1}{4}) = 1, \ y'(\frac{1}{4}) = 1. \end{cases}$$

**b)** Solve the differential equation: $y'' + y = \frac{1}{1+\sin x}$

**c)** Solve the following system of linear differential equations

$$\begin{cases} \frac{d^2x}{dt^2} + y - 1 = 0 \\ \frac{d^2y}{dt^2} + x + 1 = 0 \end{cases}$$

**Question 4.** [5,5] **a)** Find the Fourier sine series of the function $f(x) = 1 - x$, on $[0,1]$.

Deduce the value of the numerical series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$.

**b)** Consider the function: $f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 3 \\ 0, & x \ge 3 \end{cases}$

Show that $f$ can be written as a Fourier integral, sketch its graph, find its Fourier integral representation, and deduce the value of $\int_0^{\infty} \frac{1-co(3\lambda)}{\lambda^2} d\lambda$.

Question 1

ⓐ $P(t) = c\,e^{kt}, \quad P(0) = P_0, \quad P(6) = 2P_0$

Find $t$ st that $P(t) = 3P_0$ (1)

$P(t) = P_0\,e^{kt}, \quad P(6) = 2P_0 = P_0\,e^{6k}$

$2 = e^{6k} \Rightarrow \ln 2 = 6k \Rightarrow k = \frac{\ln 2}{6}$

$P(t) = P_0\,e^{\frac{\ln 2}{6}t}$ . (2)

$P(t) = 3\cancel{P_0} = \cancel{P_0}\,e^{\frac{\ln 2}{6}t} \Rightarrow 3 = e^{\frac{\ln 2}{6}t}$

$\ln 3 = \frac{\ln 2}{6}t \Rightarrow t = \dfrac{\ln 3}{\frac{\ln 2}{6}} = \dfrac{6\ln 3}{\ln 2}$ years (2)

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ⓑ $\begin{cases} x(1+y^2)\,dx + (1+x^2)^2\,dy = 0 \\ y(-1) = 0 \end{cases}$

$\frac{1}{2}\int \frac{2x\,dx}{(1+x^2)^2} + \int \frac{dy}{1+y^2} = 0 \Rightarrow \boxed{\frac{1}{2}\frac{-1}{1+x^2} + \tan^{-1}(y) = c}$ (2)

From the I.C. $y(-1) = 0 \Rightarrow (\frac{1}{2})\frac{-1}{1+1} + \tan^{-1}(0) = C \Rightarrow C = \frac{-1}{4}$ (1)

Then the unique solution of the IVP is

$\tan^{-1}(y) - \frac{1}{2(x^2+1)} + \frac{1}{4} = 0$ (1)

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Question ②

ⓐ $xy'y^3 + 2y^{-2} = x^3$, $x>0$, and $y \neq 0$ on some interval $I \subset (0,\infty)$.

$y' + \frac{2}{x}y = x^2y^3$, is B.D.E, $n=3$

$y'y^{-3} + \frac{2}{x}y^{-2} = x^2$, $u = y^{-2}$, $u' = -2y^{-3}y'$ (1)

$-\frac{u'}{2} = y^{-3}y' \Rightarrow -\frac{u'}{2} + \frac{2}{x}u \overset{-2}{=} x^2$ or

$u' - \frac{4}{x} = -2x^{\cancel{3}2}$ is linear F. order,

$\mu(x) = e^{\int -\frac{4}{x}dx} = e^{\ln x^{-4}} = x^{-4}$, $x>0$ (1)

$ux^{-4} = -2\int x^2 x^{-4}dx = -2\int \frac{dx}{x^2} = \frac{2}{x} + c$

$y^{-2}x^{-4} = \frac{2}{x} + c$

$y^{-2} = 2x^3 + cx^4$ or (2)

$\boxed{1 = y^2(2x^3 + cx^4)}$

Question (2), (b)

$$\begin{cases}(1-x^2)y'' - xy' + 9y = 0\\ y(0)=0, \quad y'(0)=3\end{cases}$$

**Solution:** $\dfrac{a_1}{a_2} = \dfrac{-x}{1-x^2} = -x\sum_0^\infty (x^2)^n = \sum_0^\infty x^{2n+1}, \quad |x|<1$

$$\frac{a_0}{a_2} = \frac{9}{1-x^2} = \sum_0^\infty 9x^{2n}, \quad |x|<1$$

1

$$y = \sum_0^\infty a_n x^n, \quad |x|<1$$

$$(1-x^2)\sum_2^\infty n(n-1)a_n x^{n-2} - x\sum_1^\infty n a_n x^{n-1} + 9\sum_0^\infty a_n x^n = 0$$

$$\sum_2^\infty n(n-1)a_n x^{n-2} - \sum_2^\infty n(n-1)a_n x^n - \sum_1^\infty n a_n x^n + \sum_0^\infty 9a_n x^n = 0$$

| | | | |
|---|---|---|---|
| $n-2=k$, $n=k+2$ | $n=k$ | $n=k$ | $n=k$ |

$$\sum_0^\infty (k+2)(k+1)a_{k+2}x^k - \sum_2^\infty k(k-1)a_k x^k - \sum_1^\infty k a_k x^k + \sum_0^\infty 9a_k x^k = 0$$

$$2a_2 + 9a_0 = 0, \quad 6a_3 - a_1 + 9a_1 = 0$$

1

$$\sum_2^\infty \left[(k+1)(k+2)a_{k+2} - k(k-1)a_k - k a_k + 9a_k\right]x^k = 0$$

$$a_{k+2} = \frac{k(k-1)+k-9}{(k+1)(k+2)}a_k = \frac{(k^2-9)a_k}{(k+1)(k+2)}$$

$$a_{k+2} = \frac{(k-3)(k+3)}{(k+1)(k+2)}a_k, \quad k\geq 0$$

2

$y(0)=0 \Rightarrow a_0=0$, $y'(0)=3 \Rightarrow a_1=3$

$$2a_2 + 9a_0 = 0 \Rightarrow a_2 = 0$$

$$6a_3 + 8a_1 = 0, \quad 6a_3 = -8a_1, \quad a_3 = \frac{-8}{\cancel{6}_2}(\cancel{3}) = -4 \qquad a_3 = -4$$

$k=2 \Rightarrow a_4 = a_6 = a_8 = \dots = 0$

$k=3 \Rightarrow a_5 = a_7 = \dots = 0$

$$\boxed{y(x) = 3x - 4x^3}$$

1

Question ③

ⓐ $\begin{cases} \ln(x+3)\, y'' + \ln(1-4x^2)\, y' + y = 2x \\ y(\tfrac{1}{4}) = 1, \quad y'(\tfrac{1}{4}) = 1. \end{cases}$

$a_2(x) = \ln(x+3)$ is cont. on $\{x \in \mathbb{R};\ x > -3\}$ 1

$a_1(x) = \ln(1-4x^2)$ is cont. on $\{x \in \mathbb{R};\ 1-4x^2\}$ 1

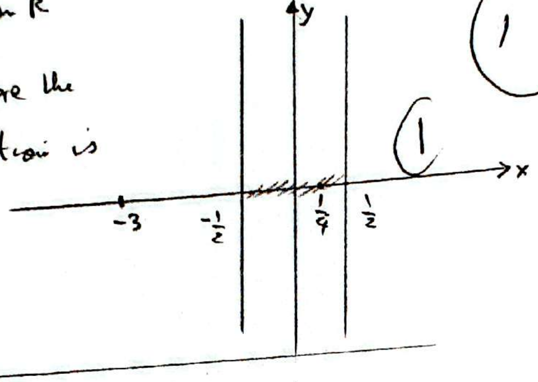
$= \{x \in \mathbb{R},\ |x| < \tfrac{1}{2}\}$

$a_0(x) = 1$, is cont. on $\mathbb{R}$.

$g(x) = 2x$ is cont. on $\mathbb{R}$ 1

The largest interval $I$ where the IVP has a unique solution is 1

$I = (-\tfrac{1}{2}, \tfrac{1}{2})$



Question ③ ⓑ $y'' + y = \dfrac{1}{1+\sin x}$ ; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

1) $y'' + y = 0$, $y = e^{mx}$, $m^2 + 1 = 0$, $m = \mp i$

$y_c = c_1 \cos x + c_2 \sin x$ $\quad y_1 = \cos x,\ y_2 = \sin x$

2) $y_p = y_1 u_1 + y_2 u_2$ s.t. $\cos x\, u_1' + \sin x\, u_2' = 0$ 1

$-\sin x\, u_1' + \cos x\, u_2' = \dfrac{1}{1+\sin x}$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$u_1' = \dfrac{\begin{vmatrix} 0 & \sin x \\ \frac{1}{1+\sin x} & \cos x \end{vmatrix}}{W} = \dfrac{-\sin x}{1+\sin x} = \dfrac{-\sin x(1-\sin x)}{(1-\sin^2 x)} = \dfrac{-\sin x(1-\sin x)}{\cos^2 x}$ 1

$= \dfrac{-\sin x}{\cos^2 x} + \tan^2 x$

$u_1 = -\int \dfrac{\sin x}{\cos^2 x}\,dx + \int (\sec^2 x - 1)\,dx$

$\boxed{u_1 = \dfrac{-1}{\cos x} + \tan x - x} = -\sec x + \tan x - x$

$u_2' = \dfrac{\begin{vmatrix} \cos x & 0 \\ \sin x & \frac{1}{1+\sin x} \end{vmatrix}}{W} = \dfrac{\cos x}{1+\sin x} \Rightarrow u_2 = \int \dfrac{\cos x}{1+\sin x}\,dx = \ln(1+\sin x)$ 2

$y_p = \cos x \left(\dfrac{-1}{\cos x} + \tan x - x\right) + \sin x \ln(1+\sin x)$

$y_p = -1 + \sin x - x\cos x + \sin x \ln(1+\sin x)$

$y = y_c + y_p = \boxed{c_1 \cos x + c_2 \sin x - 1 + \sin x - x\cos x + \sin x \ln(1+\sin x)}$

is the G. solution of the D.E

Question ③

© $\ddot{x}+y-1=0$

$\ddot{y}+x+1=0$

method

$$\begin{cases} D^2x+y=1 \\ x+D^2y=-1 \end{cases} \Rightarrow \begin{matrix} D^2x+y=1 \\ D^2x+D^4y=D^2(-1) \end{matrix} \Rightarrow \begin{matrix} \cancel{D^2x}+y=1 \\ \mp\cancel{D^2x}\mp D^4y=0 \end{matrix}$$

$y-D^4y=1$ or $y^{(4)}-y=-1$

1) $y^{(4)}-y=0$, $y=e^{mt}$, $m^4-1=(m^2-1)(m^2+1)=0$

$=(m-1)(m+1)(m^2+1)=0$

$m=1$, $m=-1$, $m=\mp i$

$y_c=c_1e^{t}+c_2e^{-t}+c_3\cos t+c_4\sin t$

2) $y_p=A$, $y_p=1$

3) $\boxed{y(t)=y_c+y_p=c_1e^{t}+c_2e^{-t}+c_3\cos t+c_4\sin t+1}$

Now $x=-\ddot{y}-1$

$\dot{y}=c_1e^{t}-c_2e^{-t}-c_3\sin t+c_4\cos t$

$\ddot{y}=c_1e^{t}+c_2e^{-t}-c_3\cos t-c_4\sin t$

$\boxed{x(t)=-c_1e^{t}-c_2e^{-t}+c_3\cos t+c_4\sin t-1}$

Method 2

$$\begin{cases} D^4x+\cancel{D^2y}=D^2(1)=0 \\ \mp\ x\mp\cancel{D^2y}=\pm1 \end{cases} \Rightarrow D^4x-x=1 \text{ or } x^{(4)}-x=1$$

1) $x^{(4)}-x=0$, $x=e^{mt}$, $m^4-1=0 \Rightarrow m=1,-1,\mp i$

$x(t)=c_1e^{t}+c_2e^{-t}+c_3\cos t+c_4\sin t$

$x_p=A$, $-A=+1 \Rightarrow A=-1$

$\boxed{x(t)=c_1e^{t}+c_2e^{-t}+c_3\cos t+c_4\sin t-1}$

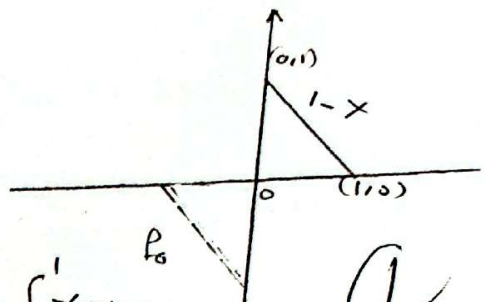
$y=-\ddot{x}+1$, $\dot{x}(t)=c_1e^{t}-c_2e^{-t}-c_3\sin t+c_4\cos t$

$\ddot{x}=c_1e^{t}+c_2e^{-t}-c_3\cos t-c_4\sin t$

$\boxed{y(t)=-c_1e^{t}-c_2e^{-t}+c_3\cos t+c_4\sin t+1}$

Question (4)

(a) $b_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx$




$$= 2\int_0^1 (1-x)\sin(n\pi x)\,dx$$

$$= 2\left[(1-x)\left(\frac{-\cos n x\pi}{n\pi}\right)\right]_0^1 - 2\int_0^1 \frac{-\cos n\pi x}{n\pi}(-dx) \quad 1$$

$$= 2\left[0-(1)\frac{(-1)}{n\pi}\right] - \left[2\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$b_n = \frac{2}{\pi n}.$$

Then $f(x) = \sum_1^\infty \frac{2}{\pi n}\sin(n\pi x) \qquad 0\le x\le 1$ 1

For $x=\frac{1}{2}$, we have $f(\frac{1}{2}) = \frac{1}{2} = \sum_1^\infty \frac{2}{\pi n}\overbrace{\sin(n\frac{\pi}{2})}^{(-1)}$ 1

$$\frac{1}{2} = \frac{2}{\pi}\sum_1^\infty \frac{1}{(2n-1)}\overbrace{\sin\left(\frac{2n-1}{2}\pi\right)}^{(-1)^{n-1}}$$

$$\boxed{\frac{\pi}{4} = \sum_1^\infty \frac{(-1)^{n-1}}{(2n-1)}} \quad 2$$

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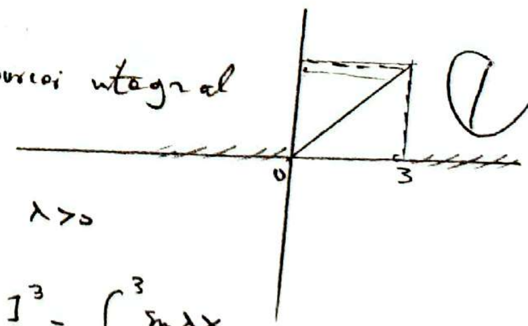
(b) $f(x) = \begin{cases} 0, & x<0 \\ x, & 0\le x\le 3 \\ 0 & x>3 \end{cases}$

f satisfies the condition of Fourier integral 1




then

$$A(\lambda) = \int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx, \quad \lambda>0$$

$$= \int_0^3 x\cos\lambda x\,dx = \left[x\frac{\sin\lambda x}{\lambda}\right]_0^3 - \int_0^3 \frac{\sin\lambda x}{\lambda}dx$$

$$= \frac{3\sin 3\lambda}{\lambda} + \left[\frac{\cos\lambda x}{\lambda^2}\right]_0^3 = \boxed{\frac{3\sin 3\lambda}{\lambda} + \frac{\cos 3\lambda - 1}{\lambda^2}} \quad 1$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = \int_0^3 x\sin\lambda x\,dx$$

$$= \left[x\left(\frac{-\cos\lambda x}{\lambda}\right)\right]_0^3 + \int_0^3 \frac{\cos\lambda x}{\lambda}dx$$

$$= \frac{-3\cos(3\lambda)}{\lambda} + \left[\frac{\sin\lambda x}{\lambda^2}\right]_0^3 = \boxed{-\frac{3\cos(3\lambda)}{\lambda} + \frac{\sin 3\lambda}{\lambda^2}} \quad 1$$

Then $f(x) = \frac{1}{\pi} \int_0^\infty \Big[ \Big( \frac{3 \sin 3\lambda}{\lambda} + \frac{\cos 3\lambda - 1}{\lambda^2} \Big) \cos \lambda x +$

$\Big( \frac{-3 \cos 3\lambda}{\lambda} + \frac{\sin 3\lambda}{\lambda^2} \Big) \sin \lambda x \Big] d\lambda$

Now, for $x = 3$, we have

$$\frac{f(3^+) + f(3^-)}{2} = \frac{3}{2} = \frac{1}{\pi} \int_0^\infty \Big[ \Big( \frac{3 \sin 3\lambda}{\lambda} + \frac{\cos 3\lambda - 1}{\lambda^2} \Big) \cos 3\lambda + \Big( \frac{-3 \cos 3\lambda}{\lambda} + \frac{\sin 3\lambda}{\lambda^2} \Big) \sin 3\lambda \Big] d\lambda$$

$$\frac{3}{2} = \frac{1}{\pi} \int_0^\infty \frac{\cos^2 3\lambda - \cos 3\lambda + \sin^2 3\lambda}{\lambda^2} d\lambda$$

$$\boxed{\frac{3\pi}{2} = \int_0^\infty \frac{1 - \cos 3\lambda}{\lambda^2} d\lambda}$$