Local extrema (القيم القصوى المحلية)

Critical points: (النقاط الحرجة)

Def: A critical point for $f(x,y)$ is a solution of:
$f_x = f_y = 0$, or a point where $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$ does not exist.

Exp: Find the critical points of:

$$f(x,y) = x^3 - 3x + y^2 - 2y + 1$$

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \\ \frac{\partial f}{\partial y} = 2y - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} 3(x^2-1) = 0 \Rightarrow \boxed{x = \pm 1} \\ 2(y-1) = 0 \Rightarrow \boxed{y = 1} \end{cases}$$

The critical points of f are: $(1,1)$ and $(-1,1)$

Exp2 $f(x,y) = x^3 + 3x + y^2 - 2y + 1$

$$\left.\begin{array}{l} f_x = 3x^2 + 3 \neq 0 \\ f_y = 2y - 2 = 0 \Rightarrow \boxed{y = 1} \end{array}\right) \Rightarrow \text{There is not critical points.}$$

Def 1 Let $f(x,y)$ be a differentiable function and let (a,b) be a critical point.

$$\left(f_x(a,b) = f_y(a,b) = 0\right)$$

We say that $f(a,b)$ is a local maximum if:

$\exists$ D: open disk with center (a,b) such that:

$$\forall (x,y) \in D \Rightarrow f(x,y) \leqslant f(a,b)$$

Similarly: $f(a,b)$ is a local minimum if:

$\exists$ open disk D centered at (a,b) such that:

$$\forall (x,y) \in D \Rightarrow f(x,y) \geqslant f(a,b).$$

Theorem: Let $f(x,y)$ be a diff. function.

Let (a,b) be a critical point of f. $\left(f_x(a,b) = f_y(a,b) = 0\right)$

Define $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$

(a) If $D(a,b) > 0$, then $f(a,b)$ is a local max. when $f_{xx}(a,b) < 0$.
and it is a local min when $f_{xx}(a,b) > 0$

(b) If $D(a,b) < 0$, then $f(a,b)$ is not a local extremum.
In this case, we say that $(a,b,f(a,b))$ is a saddle point
نقطة سرج

(c) If $D(a,b) = 0$ (Nothing can be deduced)

Example 1 $f(x,y) = x^3 + 3xy^2 - 3x + 1$

Find the local extrema and saddle points for f.

Solution: Critical points:

$$\begin{cases} f_x = 3x^2 + 3y^2 - 3 = 0 & ① \\ f_y = 6xy = 0 \; ② \Rightarrow x = 0 \text{ or } y = 0 \end{cases}$$

If $x = 0$: Into ①, we get: $3y^2 = 3 \quad y^2 = 1 \Leftrightarrow y = \pm 1$

If $y = 0$ $\quad 3x^2 = 3 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$

The critical points are: $(0,1), (0,-1), (1,0), (-1,0)$

$$\begin{cases} f_{xx} = 6x \\ f_{xy} = 6y \\ f_{yy} = 6x \end{cases} \qquad D = f_{xx} \cdot f_{yy} - f_{xy}^2 = 36x^2 - 36y^2$$

$D(0,1) = -36 < 0 \Rightarrow (0, 1, \overset{f(0,1)}{\overset{\shortparallel}{1}})$ is a saddle point.

$D(0,-1) = -36 < 0 \Rightarrow (0, -1, 1)$ " " "

$D(1,0) = 36 > 0$, $f_{xx}(1,0) = 6 > 0$

$\Rightarrow f(1,0) = -1$ is a local minimum.

$D(-1,0) = 36 > 0$, $f_{xx}(-1,0) = -6 < 0$

$\Rightarrow f(-1,0) = 3$ is a local max.

Example 2 $f(x,y) = x^3 + y^3 - 6xy + 2$

Critical points:

$$\begin{cases} f_x = 3x^2 - 6y = 0 \\ f_y = 3y^2 - 6x = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 = 2y & ① \\ y^2 = 2x & ② \end{cases}$$

$$① \Leftrightarrow x^4 = 4y^2 \overset{\text{by } ②}{\Leftrightarrow} x^4 = 4 \cdot 2x$$

$$\Leftrightarrow x^4 = 8x \Leftrightarrow x^4 - 8x = 0 \Leftrightarrow x(x^3 - 8) = 0$$

$$\Rightarrow \underline{x = 0} \text{ or } \underline{x = 2}$$

$$① \Downarrow \qquad \Downarrow ①$$

$$y = 0 \qquad y = 2$$

The critical points are $(0,0)$ and $(2,2)$.

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -6$

$$D = 36xy - (-6)^2 = 36xy - 36$$

$D(0,0) = -36 \Rightarrow (0,0,2)$ is a saddle point.

$D(2,2) = 36 \times 3 > 0$ and $f_{xx}(2,2) = 12 > 0$

$\Rightarrow f(2,2) = -6$ is a local min.

Global extrema:

Theorem: Any continuous function f on a bounded and closed domain D attains its max and min on D.

Example: Find the extremas of the function

$$f(x,y) = x^2 - 4x + y^2 - 2y + 1$$

on the region $D = \{(x,y) : 1 \leq x \leq 3 \quad 0 \leq y \leq x\}$

Inside D: $\begin{cases} f_x = 2x - 4 = 0 \Rightarrow x = 2 \\ f_y = 2y - 2 = 0 \Rightarrow y = 1 \end{cases}$

$(2,1) \in D \Rightarrow \boxed{f(2,1) = -4}$

S_1: $y = 0 \quad 1 \leq x \leq 3 \quad f = x^2 - 4x + 1$

$f' = 2x - 4 = 0 \Leftrightarrow \underline{x = 2}$ ✓ $\quad \boxed{f(2,0) = -3}$

S_2: $x = 3 \quad 0 \leq y \leq 3$

$f = y^2 - 2y - 2 \quad \boxed{f(3,1) = -3}$

$f' = 2y - 2 = 0 \Rightarrow y = 1$

S_3: $x=1 \quad 0 \leq y \leq 1$

$$f = y^2 - 2y - 2 \qquad \boxed{f(1,1) = -3}$$

$$f' = 2y - 2 = 0 \Rightarrow y = 1$$

S_4: $y = x \quad 1 \leq x \leq 3$

$$f = x^2 - 4x + x^2 - 2x + 1 = 2x^2 - 6x + 1$$

$$f' = 4x - 6 = 0 \Rightarrow x = 3/2 \qquad \boxed{f(3/2, 3/2) = -7/2}$$

(x,y)	$(2,1)$	$(2,0)$	$(3,1)$	$(1,1)$	$(3/2,3/2)$	$(1,0)$	$(3,0)$	$\cancel{(1,1)}$	$(3,3)$
$f(x,y)$	-4	-3	-3	-3	$-7/2$	-2	-2	$\cancel{-3}$	1

The maximum is 1

The minimum is -4