Exp4 $\quad f(x,y) = \begin{cases} \dfrac{x^3y+y^4}{x^2+y^2} \\ 0 & (x,y)=(0,0) \end{cases}$

* f continuous at $(0,0)$ (ST)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim \frac{\frac{0+h^4}{0+h^2}-0}{h}$$

$$= \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left|f(\Delta x,\Delta y) - \overbrace{f(0,0)}^{=0} - \overbrace{f_x(0,0)}^{=0}\Delta x - \overbrace{f_y(0,0)}^{=0}\cdot\Delta y\right|}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{|\Delta x^3\Delta y+\Delta y^4|}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}} = 0 \quad (ST)$$

$$0 \leq \frac{|\Delta x^3\Delta y+\Delta y^4|}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}} \leq \frac{|\Delta x^3\Delta y|}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}} + \frac{\Delta y^4}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}}$$

$$= \frac{\Delta x^2 \, |\Delta x| \cdot |\Delta y|}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}} + \frac{\Delta y^2\, |\Delta y| \cdot |\Delta y|}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}}$$

≤ 1 $\quad \leq 1$ $\quad \leq 1$ $\quad \leq 1$

$$\leq 2|\Delta y|$$

The Chain rule:

Theorem: If $w = f(x,y,z)$ is a differentiable function with $x = x(t)$, $y = y(t)$, $z = z(t)$, then:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

or $\quad w'(t) = f_x \cdot x' + f_y \cdot y' + f_z \cdot z'$

Example If $w = x^2 y + \sin(x+y^2)$ and $x = e^t$, $y = t^2 - 1$. Find $\frac{dw}{dt}(0)$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = \left[2xy + 1\cdot\cos(x+y^2)\right]\cdot e^t + \left(x^2 + 2y\cos(x+y^2)\right)\cdot 2t$$

$$= \left[2e^t(t^2-1) + \cos\left(e^t + (t^2-1)^2\right)\right]e^t + 2t\left(e^{2t} + 2\,(t^2-1)\cos\left(e^t + (t^2-1)^2\right)\right)$$

$$\frac{dw}{dt}(0) = (-2 + \cos 2) + 0 = -2 + \cos 2.$$

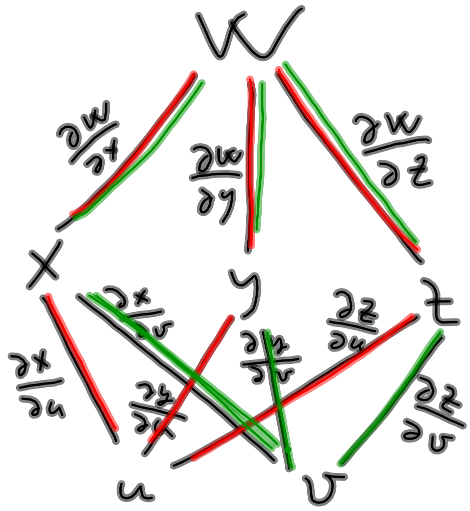
Theorem: let $w = f(x, y, z)$ be a differentiable function such that:

$x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$,

and x, y, z are differentiable. Then,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$$



Example If $w = \ln(xyz + 1)$, with $x = u + v$, $y = u - v$, $z = uv$. Use the Chain rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \left(\frac{yz}{xyz+1}\right)(1) + \left(\frac{xz}{xyz+1}\right)(1) + \left(\frac{xy}{xyz+1}\right)(v)$$

$$= \frac{1}{(u+v)(u-v)uv+1}\Big[(u-v)uv + (u+v)uv + (u+v)(u-v)v\Big]$$

$$\frac{\partial w}{\partial v} = \frac{1}{(u+v)(u-v)uv+1}\Big[(u-v)uv - (u+v)uv + (u+v)(u-v)1$$

Implicit differentiation: (الاشتقاق الضمني)

Theorem 1 If y is differentiable function in x defined by the equation $F(x,y)=0$, where F is differentiable. Then:

$$y' = -\frac{F_x}{F_y}$$

Exp: If $xy+1 = x^4 - y^3$, find y'.

Set: $F(x,y) = xy+1-x^4+y^3$

$$F(x,y)=0 \Rightarrow y' = -\frac{F_x}{F_y} = -\frac{y-4x^3}{x+3y^2}$$

Find $y'(0)$. $0+1 = 0-y^3 \Rightarrow y=-1$

$y'(0) = -\frac{-1+0}{\ } = 1/$

<u>Theorem 2</u> If z is a differentiable function in (x,y) defined by: $F(x,y,z)=0$ (F is also differentiable), then:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

<u>Exp:</u> Find $\frac{\partial z}{\partial x}(1,0)$ and $\frac{\partial z}{\partial y}(1,0)$ if $x^2+y^3+z = xyz^4+1$

put $F(x,y,z) = x^2+y^3+z-xyz^4-1 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x-yz^4}{1-4xyz^3}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2-xz^4}{1-4xyz^3}$$

$\underline{x=1}$, $\underline{y=0} \Rightarrow z = ?$

$1+0+z = 0+1$

$\Rightarrow \underline{z=0}$

$$\Rightarrow \frac{\partial z}{\partial x}(1,0) = -\frac{2}{1} = -2.$$

$$\frac{\partial z}{\partial y}(1,0) = -\frac{0}{1} = 0$$