Partial differentiation:

Def. Let f be a function of two variables x, y.

* The partial derivative of f at (a,b) in the x-direction:

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

* The partial derivative of f at (a,b) in the y-direction:

$$\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Examples:

* To compute $\frac{\partial f}{\partial x}$ we fix y and differentiate f with respect to x.
* Similarly for $\frac{\partial f}{\partial y}$.

1. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for $f(x,y) = x^2 + y^3 + 1$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 3y^2$$

2. $f(x,y) = x^3 + y^4 + xy^2 + 1$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial f}{\partial y} = 4y^3 + 2xy$$

$\boxed{3}$ $f(x,y) = \sin\left(\sqrt{x^2+xy}\right)$

$$\frac{\partial f}{\partial x} = (2x+y)\cdot\frac{1}{2\sqrt{x^2+xy}}\cdot\cos\left(\sqrt{x^2+xy}\right)$$

$$\begin{tikzcd}
(x,y) \arrow[r] \arrow[drr] & x^2+xy \arrow[r] & \sqrt{x^2+xy} \arrow[d] \\
 & & \sin\left(\sqrt{x^2+xy}\right)
\end{tikzcd}$$

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{x^2+xy}}\cos\left(\sqrt{x^2+xy}\right)$$

$\boxed{4}$ $f(x,y) = \begin{cases} \dfrac{x^2-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Compute $\frac{\partial f}{\partial x}(1,0)$, $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$.

$\underline{\text{at } (x,y) \neq (0,0)}$

$$\frac{\partial f}{\partial x} = \frac{2x(x^2+y^2) - 2x(x^2-y^3)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(1,0) = \frac{2(1+0) - 2(1-0)}{(1+0)^2} = 0.$$

At $(0,0)$: (The defintion of $\frac{\partial f}{\partial x}$)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^2-0}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{1}{h} = \pm\infty$$

$$\boxed{\frac{\partial f}{\partial x}(0,0) \text{ does not exist}}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0-h^3}{0^2+h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}} = -1$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y}(0,0) = -1}$$

Def. If $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are differentiable, then we define:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

* Similarly, we can define the partial derivatives of order $n = 3, 4, \cdots$

$$\frac{\partial^3 f}{\partial x^2 \partial y}, \frac{\partial^3 f}{\partial x \partial y^2}, \cdots, \frac{\partial^3 f}{\partial x \partial z \partial y}, \cdots \text{etc}$$

(or f_{xxy} , f_{xyy} , f_{xzy})

Exp: $f(x,y) = x^3 y^2$

$\frac{\partial^2 f}{\partial x^2} = ?$ $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$?

$$\frac{\partial f}{\partial x} = 3x^2y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy^2 \qquad \boxed{\frac{\partial^2 f}{\partial y \partial x} = 6x^2 y}$$

$$\frac{\partial f}{\partial y} = 2x^3 y \Rightarrow \boxed{\frac{\partial^2 f}{\partial x \partial y} = 6x^2 y}, \quad \frac{\partial^2 f}{\partial y^2} = 2x^3$$

* In general $\frac{\partial^2 f}{\partial x \partial y}$ need not be equal to $\frac{\partial^2 f}{\partial y \partial x}$. But In Math 201, we will consider only functions satisfying

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Exp: $f(x,y) = x^3 y^2$

$f_{xyxy} = 12x$

$f_x = 3x^2y^2$
$f_{xx} = 6xy^2$
$f_{yxx} = 12xy$
$f_{yyxx} = 12x$

Exp: $f(x,y) = \ln(x^2y^2z^3)$

$f_{zx} = ?$ $\quad f_x = \frac{2\cancel{x}\cancel{y^2}\cancel{z^3}}{x^{\cancel{2}}\cancel{y^2}\cancel{z^3}} = \frac{2}{x}$

$\Rightarrow \boxed{f_{zx} = 0}$

$f_z = \frac{3\cancel{x^2}\cancel{y^2}\cancel{z^2}}{\cancel{x^2}\cancel{y^2}z^{\cancel{3}}} = \frac{3}{z} \Rightarrow \boxed{f_{xz} = 0}$

Differentiability: (قابلية التفاضل)

Def: let $f(x,y)$ be a continuous map at (a,b), and $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ exist.

we say that f is differentiable at (a,b) if:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\left| f(a+\Delta x, b+\Delta y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y \right|}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

Examples:

$\boxed{1}$ Any polynomial function is differentiable

$\boxed{2}$ study the differentiability of f at $(0,0)$:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

* f is not continuous at $(0,0)$: * $f(0,0) = 0$

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (by paths)

$\Rightarrow$ f is not differentiable at $(0,0)$.

$\boxed{3}$ $$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

* f is continuous (by ST)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{|f(0+\Delta x,\Delta y) - f(0,0) - f_x(0,0)\cdot\Delta x - f_y(0,0)\cdot\Delta y|}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\left|\frac{\Delta x^2\Delta y}{\Delta x^2+\Delta y^2} - 0 - 0\cdot\Delta x - 0\cdot\Delta y\right|}{\sqrt{\Delta x^2+\Delta y^2}} \qquad \sqrt{a^2} = |a|$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x^2|\Delta y|}{(\Delta x^2+\Delta y^2)\sqrt{\Delta x^2+\Delta y^2}} \qquad \sqrt{2\Delta x^2} = \sqrt{2}\,|\Delta x|$$

$\underline{\text{on } \Delta y = \Delta x}$ $$\lim_{\Delta x\to 0} \frac{\cancel{\Delta x^2}\cdot\cancel{|\Delta x|}}{2\cancel{\Delta x^2}\cdot\sqrt{2}\,\cancel{|\Delta x|}} = \frac{1}{2\sqrt{2}} \neq 0$$

$\Rightarrow$ f is not differentiable at $(0,0)$.