<u>Examples:</u> $\boxed{1}$ Find the domain of the function:

$$f(x,y) = \sqrt{1-x^2-y^2}$$

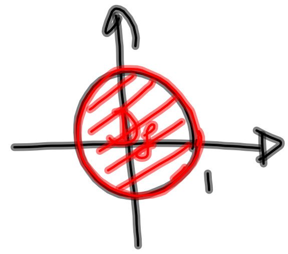
$$D_f = \{(x,y) : \sqrt{1-x^2-y^2} \text{ exists}\}$$

$$= \{(x,y) \quad 1-x^2-y^2 \geq 0\}$$

$1-x^2-y^2=0$?

$x^2+y^2=1$

unit circle



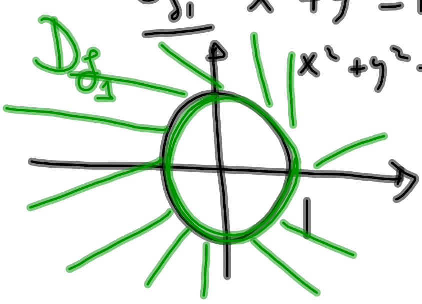
$\boxed{2}$ $$f(x,y) = \underbrace{\sqrt{x^2+y^2-1}}_{f_1} + \underbrace{\sqrt{4-x^2-y^2}}_{f_2}$$

$$D_{f_1} = \{(x,y) : x^2+y^2-1 \geq 0\}$$

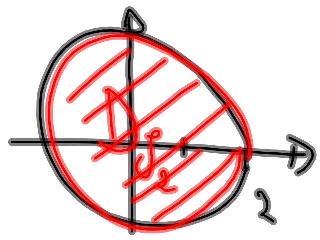
$$D_{f_2} = \{(x,y) : 4-x^2-y^2 \geq 0\}$$

<u>D_{f_1}</u> $x^2+y^2-1=0$

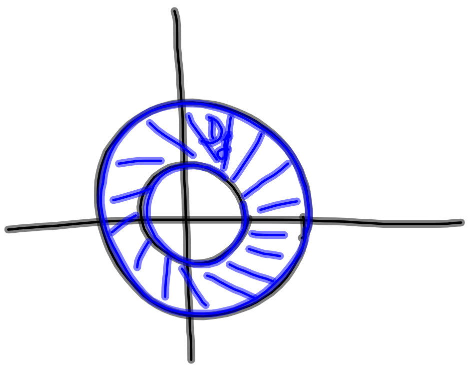
$x^2+y^2=1$



<u>D_{f_2}</u>



<u>D_f :</u>



Limits:

Def: Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function.
Let $(a,b) \in D$. We say that $f(x,y)$ tends to the real number L when (x,y) tends to (a,b) if: $\forall \varepsilon > 0, \exists \delta > 0 : \sqrt{(x-a)^2+(y-b)^2} < \delta$

$$\Rightarrow |f(x,y) - L| < \varepsilon.$$

and we write: $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

Theorem: (Sandwich Theorem)
Let f, g, h be three functions defined on $D \subset \mathbb{R}^2$ such that $g(x,y) \leq f(x,y) \leq h(x,y)$ $\forall (x,y) \in D$.
Let $(a,b) \in D$. If $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = \lim\limits_{(x,y)\to(a,b)} h(x,y) = L$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

Examples:

1) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2} = 0$

$$0 \leq \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1}|y| \leq 1.|y| = |y|$$

Since $\lim\limits_{(x,y)\to(0,0)} 0 = \lim\limits_{(x,y)\to(0,0)} |y| = 0$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2} = 0$ (by applying the ST)

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy \sin\sqrt{x^2+y^2}}{x^2+y^2} = 0$

$$0 \leq \left| \underbrace{\frac{xy}{x^2+y^2}}_{\leq} \sin\sqrt{x^2+y^2} \right| \leq 1.\sin\sqrt{x^2+y^2}$$

$\downarrow$ ST

$0 \qquad 0 \qquad 0$

④ $$\lim_{(x,y)\to(0,0)} \frac{\sin x^2 y}{\sqrt{x^2+y^2}} = 0$$

$$0 \leq \frac{|\sin x^2 y|}{\sqrt{x^2+y^2}} \leq \frac{x^2 \overbrace{\frac{|y|}{\sqrt{x^2+y^2}}}^{\leq 1}}{} \leq x^2$$

Since $\lim\limits_{(x,y)\to(0,0)} 0 = \lim\limits_{(x,y)\to(0,0)} x^2 = 0$, then $\lim\limits_{(x,y)\to(0,0)} \frac{\sin x^2 y}{\sqrt{x^2+y^2}} = 0$

⑤ $$\lim_{(x,y)\to(0,0)} xy \sin\left(\frac{1}{x^2+y^2}\right) = ?$$

$$0 \leq \left|xy \overbrace{\sin\left(\frac{1}{x^2+y^2}\right)}^{\leq 1}\right| \leq |xy| \cdot 1$$

Since $\lim\limits_{(x,y)\to(0,0)} 0 = \lim\limits_{(x,y)\to(0,0)} |xy| = 0$,

then $\lim\limits_{(x,y)\to(0,0)} xy \sin\left(\frac{1}{x^2+y^2}\right) = 0$

Paths Theorem:

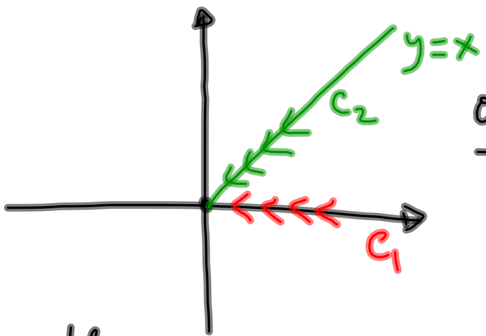
If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L_1$ on the path C_1 and

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L_2$ on the path C_2, and $L_1 \neq L_2$,

then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist

(C_1 and C_2 are paths through (a,b))

__Exp1__ prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.



__on C_1:__ $y=0$

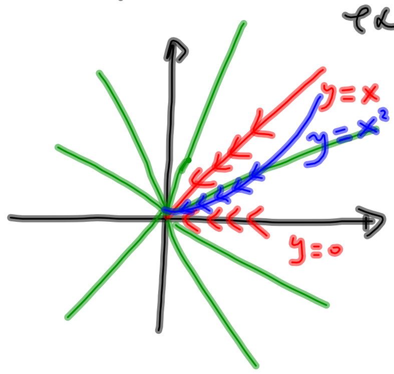
$$\lim_{x\to 0} \frac{x \cdot 0}{x^2+0^2} = 0$$

__on the path C_2:__ $y=x$

$$\lim_{x\to 0} \frac{x \cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

$$\Longrightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist.}$$

Exp 2 Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$ does not exist.



On $y=0$: $\lim\limits_{x\to 0} \dfrac{x^2\cdot 0}{x^4+0^2} = 0$

On $y=x$: $\lim\limits_{x\to 0} \dfrac{x^2\cdot x}{x^4+x^2}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x^2}\cdot x}{\cancel{x^2}(x^2+1)} = 0$

On $y=mx$: $\lim\limits_{x\to 0} \dfrac{x^2\cdot mx}{x^4+m^2x^2} = \lim\limits_{x\to 0} \dfrac{\cancel{x^2}\cdot mx}{\cancel{x^2}(x^2+m^2)} = 0$

On $y=x^2$:

$\lim\limits_{x\to 0} \dfrac{x^2\cdot x^2}{x^4+x^4} = \lim\limits_{x\to 0} \dfrac{\cancel{x^4}}{2\cancel{x^4}} = \dfrac{1}{2}$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$ does not exist.

Functions of three variables:

Exp 1 $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+3y^2+z^2}$

$|xy| \leqslant x^2+y^2 \leqslant x^2+3y^2 \leqslant x^2+3y^2+z^2$

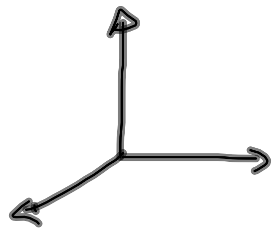
$\Rightarrow \dfrac{|xy|}{x^2+3y^2+z^2} \leqslant 1$

$0 \leqslant \dfrac{|xyz|}{x^2+3y^2+z^2} \leqslant |z|$ ($\leqslant 1$)

Since $\lim\limits_{(x,y,z)\to(0,0,0)} 0 = \lim\limits_{(x,y,z)\to(0,0,0)} |z| = 0$, then

$\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+3y^2+z^2} = 0$

Exp 2 $$\lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2+3y^2+z^2}$$



$\overrightarrow{OX}$ axis: $y = z = 0$

$$\lim_{x\to 0} \frac{x \cdot 0}{x^2+0+0} = 0$$

on $x = y = z$ $$\lim_{x\to 0} \frac{x^2}{x^2+3x^2+x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{5\cancel{x^2}} = \frac{1}{5}$$

$$\Rightarrow \lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2+3y^2+z^2} \text{ does not exist.}$$

Continuous maps:

Def. We say that $f(x,y)$ is continuous at (a,b) if:

(a) $f(a,b)$ exists.

(b) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(c) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$