Triple integrals:

If $f: B \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$, with

B the box: $B = [a,b] \times [c,d] \times [r,s]$

Divide $[a,b]$ into ℓ subintervals of width $\Delta x = \frac{b-a}{\ell}$

Similarly, divide $[c,d]$ into m subintervals of width $\Delta y = \frac{d-c}{m}$

and divide $[r,s]$ into n subintervals of width $\Delta z = \frac{s-r}{n}$

In each sub-box $[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, choose a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$. Set $\Delta V = \Delta x \Delta y \Delta z$

Def: The triple integral of f over the box B is:

$$\iiint_B f(x,y,z)\,dV = \lim_{\ell,m,n \to +\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

if this limit exists.

Fubini's Theorem for triple integrals:

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz \qquad B: \text{box}$$

$$= \int_c^d \int_a^b \int_r^s f(x,y,z)\,dz\,dx\,dy$$

$\vdots$

Example Evaluate $I = \iiint_B 12xyz^2\,dV$, where
$B = [0,1]^3 = \{(x,y,z) : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$

Solution:

$$I = \int_0^1\int_0^1\int_0^1 12xyz^2\,dz\,dy\,dx$$
$$= \int_0^1\int_0^1 12xy\frac{z^3}{3}\Big|_{z=0}^{z=1}\,dy\,dx$$
$$= \int_0^1\int_0^1 4xy\,dy\,dx$$
$$= \int_0^1 4x\frac{y^2}{2}\Big|_{y=0}^{y=1}\,dx$$
$$= \int_0^1 2x\,dx$$
$$= x^2\Big|_0^1$$
$$= 1$$

Theorem: If $E = \left\{(x,y,z) : \begin{array}{c} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \\ h_1(x,y) \leq z \leq h_2(x,y) \end{array}\right\}$,

then: $$\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

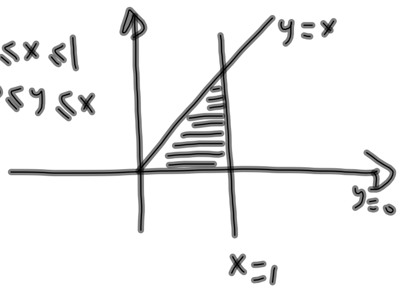
Example Evaluate the integral

$$I = \iiint_E z\,dV,$$ where E is the solid in the first octant bounded by the surface $z = 12xy$ and the planes $y = x$ and $x = 1$.

Solution: First octant: $x \geq 0$, $y \geq 0$, $z \geq 0$

$0 \leq z \leq 12xy$
$0 \leq y \leq x$
$0 \leq x \leq 1$

$0 \leq x \leq 1$
$0 \leq y \leq x$



$$I = \int_0^1 \int_0^x \int_0^{12xy} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^x \frac{z^2}{2} \Big|_0^{12xy} dy \, dx$$

$$= \int_0^1 \int_0^x 72 x^2 y^2 \, dy \, dx$$

$$= \int_0^1 72 x^2 \frac{y^3}{3} \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 24 x^2 \cdot x^3 \, dx$$

$$= \int_0^1 24 x^5 \, dx$$

$$= 24 \cdot \frac{x^6}{6} \Big|_0^1$$

$$= 4.$$

Theorem: The volume of the solid E is:

$$V(E) = \iiint_E dV$$

Example: Find the volume of the solid bounded by the coordinate planes and the plane $2x + y + z = 4$.

Solution: Coordinate planes:

xy-plane : $z = 0$

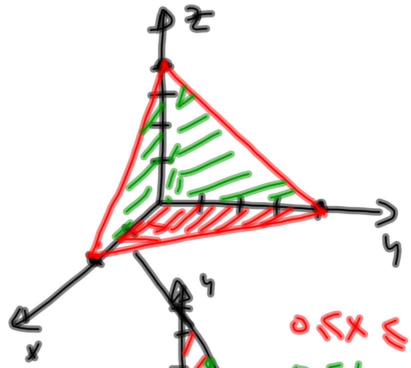
xz-plane : $y = 0$

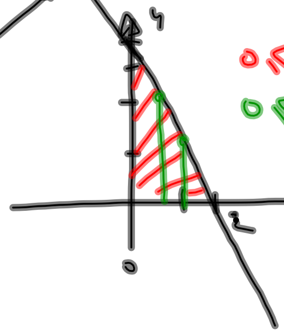
yz-plane : $x = 0$

The solid is: a tetrahedron

$$2x+y+z=4$$

$$\begin{cases} 0 \leq z \leq 4-2x-y \\ 0 \leq y \leq 4-2x \\ 0 \leq x \leq 2 \end{cases}$$





$$V=\int_0^2\int_0^{4-2x}\int_0^{4-2x-y} dz\,dy\,dx$$

$$=\int_0^2\int_0^{4-2x}(4-2x-y)\,dy\,dx$$

$$=\int_0^2 4y-2xy-\frac{y^2}{2}\Big|_{y=0}^{y=4-2x} dx$$

$$=\int_0^2\left[4(4-2x)-2x(4-2x)-\frac{(4-2x)^2}{2}\right]dx$$

$$=\int_0^2(4-2x)\left(4-2x-\frac{4-2x}{2}\right)dx$$

$$=\int_0^2(4-2x)(4-2x-2+x)\,dx$$

$$=\int_0^2(4-2x)(2-x)\,dx$$

$$=\int_0^2(8-8x+2x^2)\,dx$$

$$=8x-4x^2+\frac{2}{3}x^3\Big|_0^2$$

$$=16-16+\frac{2}{3}\cdot 8$$

$$=\frac{16}{3}.$$

Cylindrical coordinats:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad (r, \theta, z)$$

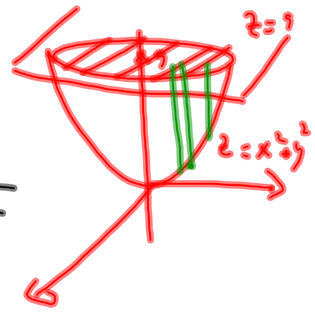
$$\iiint_Q f(x,y,z)\,dV = \iiint_Q f(r\cos\theta, r\sin\theta, z)\, r\, dz\,dr\,d\theta$$

Example: Find the volume of the solid bounded by the hyperboloid $z = x^2+y^2$ and $z = 9$.

$$V = \iiint 1 \cdot dV \qquad x^2+y^2 \leq z \leq 9$$

$$-3 \leq x \leq 3$$

$$-\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$$



$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2}^{9} dz\,dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9-x^2-y^2)\,dy\,dx$$

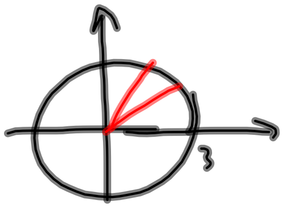
$$= \int_{-3}^{3} 9y - x^2y - \frac{y^3}{3}\Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

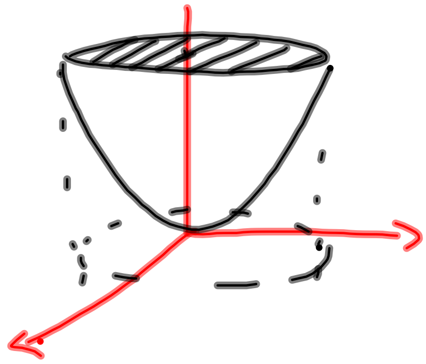
$$= \int_{-3}^{3} \left[2(9-x^2)\sqrt{9-x^2} - \frac{2}{3}(9-x^2)^{3/2}\right] dx$$

$$= \frac{1}{3}\int_{-3}^{3} (9-x^2)^{3/2}\,dx \quad : \text{We need trigonometric Substitution}$$

Solution:

$r^2 \leq z \leq 9$





In (x,y)-plane:

$0 \leq r \leq 3$

$0 \leq \theta \leq 2\pi$

$$V = \int_0^{2\pi}\int_0^3\int_{r^2}^9 r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r\,z\Big|_{z=r^2}^{z=9} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta$$

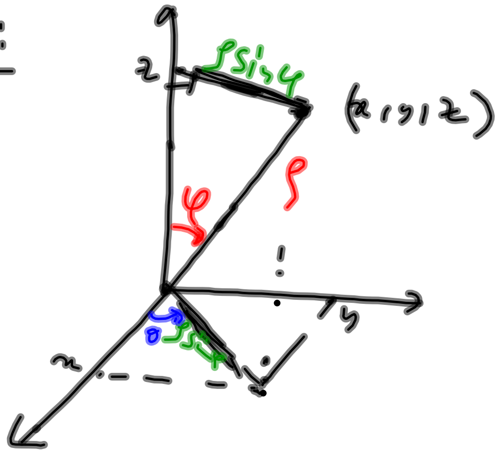
$$= \int_0^{2\pi}\int_0^3 (9r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{9r^2}{2} - \frac{r^4}{4}\Big|_0^3 d\theta$$

$$= \int_0^{2\pi}\left(\frac{81}{2}-\frac{81}{4}\right)d\theta$$

$$= \int_0^{2\pi}\frac{81}{4}\,d\theta = \frac{81\pi}{2}.$$

Spherical coordinates:



$$\cos\varphi = \frac{z}{\rho}$$

$$\Rightarrow \boxed{z = \rho\cos\varphi}$$

$$\sin\varphi = \frac{\text{المقابل}}{\text{الوتر}} = \frac{r}{\rho}$$

$$\boxed{r = \rho\sin\varphi}$$

$$\boxed{\begin{aligned} x &= r\cos\theta = \rho\sin\varphi \cdot \cos\theta \\ y &= r\sin\theta = \rho\sin\varphi \cdot \sin\theta \end{aligned}}$$

$$x^2+y^2+z^2 = \rho^2\sin^2\varphi \cdot \cos^2\theta + \rho^2\sin^2\varphi \cdot \sin^2\theta + \rho^2\cos^2\varphi$$

$$= \rho^2\sin^2\varphi\,(\underbrace{\cos^2\theta+\sin^2\theta}_{=1}) + \rho^2\cos^2\varphi$$

$$= \rho^2\sin^2\varphi + \rho^2\cos^2\varphi$$

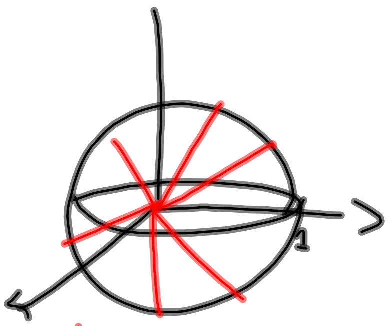
$$= \rho^2(\underbrace{\sin^2\varphi+\cos^2\varphi}_{=1})$$

$$= \rho^2$$

$$x^2+y^2+z^2 = 1$$

$$\rho^2 = 1$$

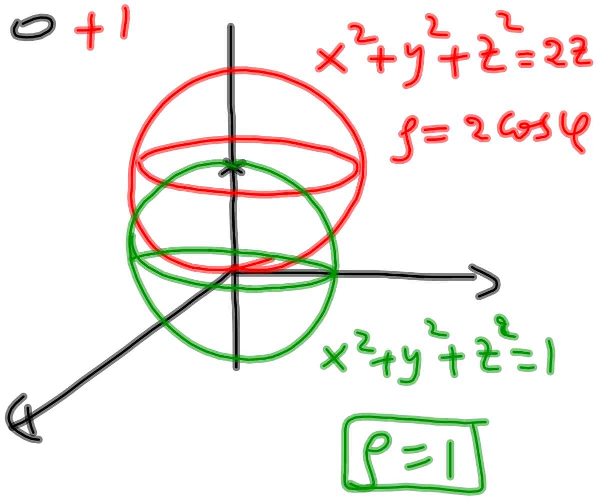
$$\boxed{\rho = 1}$$ Sphere of center (0,0,0) and radius 1.



② $x^2+y^2+z^2=2z$

$$x^2+y^2+\underbrace{z^2-2z+1}=0+1$$

$$x^2+y^2+(z-1)^2=1$$



$x^2+y^2+z^2=2z$

$$\rho^{\cancel{2}}=2\cancel{\rho}\cos\varphi$$

$$\boxed{\rho=2\cos\varphi}$$

$$\iiint_Q f(x,y,z)\,dV=\iiint_Q f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d$$

Exp1 Find the volume of the solid inside both the cone $z=\sqrt{3(x^2+y^2)}$ and the sphere $x^2+y^2+z^2=4$.

$0\le\rho\le 2$

$0\le\varphi\le\pi/6$

$0\le\theta\le 2\pi$

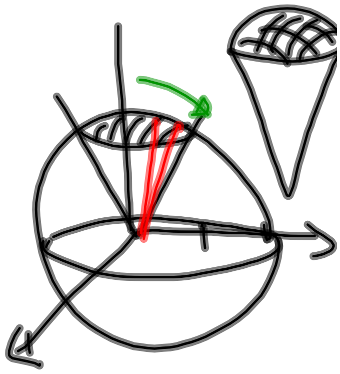
$z=\rho\cos\varphi$

$r=\rho\sin\varphi$

$z=\sqrt{3(x^2+y^2)}=\sqrt{3}\cdot r$

$\cancel{\rho}\cos\varphi=\sqrt{3}\cancel{\rho}\sin\varphi$

$\Rightarrow \tan\varphi=\frac{1}{\sqrt{3}}=\frac{1/2}{\sqrt{3}/2}\Rightarrow \boxed{\varphi=\pi/6}$



$r^2=x^2+y^2$

$$V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\pi/6} \sin\varphi \cdot \frac{\rho^3}{3}\Big|_0^2 \, d\varphi$$

$$= \frac{16\pi}{3}\int_0^{\pi/6} \sin\varphi \, d\varphi$$

$$= -\frac{16\pi}{3} \cos\varphi \Big|_0^{\pi/6}$$

$$= -\frac{16\pi}{3}\left(\frac{\sqrt{3}}{2} - 1\right)$$

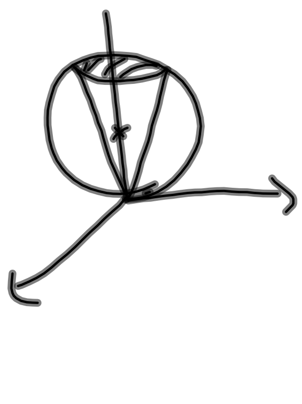
$$= \frac{-16\pi}{3}\left(\frac{\sqrt{3}-2}{2}\right)$$

$$= \frac{8\pi}{3}\left(2-\sqrt{3}\right)$$

Exp2 Find the volume of the solid inside the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 2z$

Sphere: $\rho^2 = 2\rho\cos\varphi$

$$\Rightarrow \boxed{\rho = 2\cos\varphi}$$



$$\Rightarrow \quad 0 \le \rho \le 2\cos\varphi$$
$$0 \le \varphi \le \pi/4$$
$$0 \le \theta \le 2\pi$$

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \int_0^{\pi/4} \frac{\rho^3}{3}\Big|_0^{2\cos\varphi} \cdot \sin\varphi \, d\varphi$$

$$= \frac{16\pi}{3}\int_0^{\pi/4} \cos^3\varphi \cdot \sin\varphi \, d\varphi$$

$$= -\frac{16\pi}{3} \frac{\cos^4\varphi}{4}\Big|_0^{\pi/4}$$

$$= \frac{-4\pi}{3}\left(\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right)$$

$$= \frac{-4\pi}{3}\left(\frac{1}{4} - 1\right)$$

$$= -\frac{4}{3}\pi \cdot \left(-\frac{3}{4}\right)$$

$$= \pi$$

$$z = \rho\cos\varphi$$
$$r = \rho\sin\varphi$$
$$x^2 + y^2 = r^2$$

$$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

$$\cancel{\rho}\cos\varphi = \cancel{\rho}\sin\varphi$$

$$\tan\varphi = 1$$

$$\Rightarrow \boxed{\varphi = \pi/4}$$

$$u = \cos\varphi$$
$$du = -\sin\varphi \, d\varphi$$
$$-\int u^3 du = -\frac{u^4}{4}$$