Double integrals:

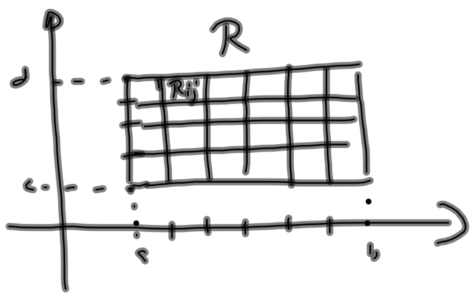
Let R be the rectangle $R=[a,b]\times[c,d]$,

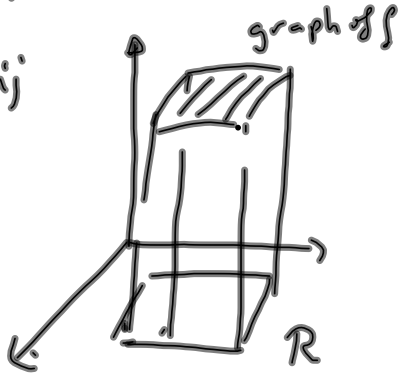
and $f: R \longrightarrow \mathbb{R}$: continuous.
$(x,y) \longmapsto f(x,y)$

Suppose $f(x,y) \geq 0$

The goal is to compute the volume of the solid that lies above R and below the graph of f.

Divide R into subrectangles: R_{ij}





This, by dividing $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of the same width $\Delta x = \frac{b-a}{m}$, and dividing $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of the same width $\Delta y = \frac{d-c}{n}$.

So, we obtain $n\times m$ subrectangles R_{ij} having the same area $\Delta A = \Delta x \cdot \Delta y = \frac{(b-a)(d-c)}{nm}$

In each R_{ij}, choose a sample point (x^*_{ij}, y^*_{ij}).

Write (Rieaman Sum): $\sum_{i,j=1}^{m,n} f(x^*_{ij}, y^*_{ij})\Delta A$

Definition: The double integral of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{\substack{n\to+\infty \\ m\to+\infty}} \sum_{i,j=1}^{m,n} f(x^*_{ij}, y^*_{ij})\Delta A$$

Remark: If $f(x,y) \geq 0$, then the double integral $\iint_R f(x,y)\,dA$ is equal to the volume of the solid above the rectangle R and below the graph of f:

$$V = \iint_R f(x,y)\,dA$$

Iterated integrals:

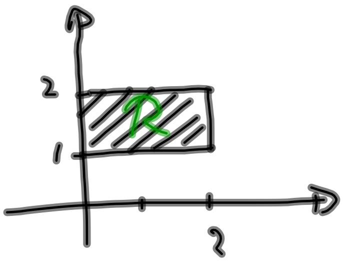
Theorem (Fubini's theorem)

If f is a continuous map defined on the rectangle $R = [a,b] \times [c,d] = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Examples: 1) Evaluate $I = \iint_R xy^2\,dA$, where

$R = [0,2] \times [1,2]$



$$I = \int_0^2 \left(\int_1^2 x y^2\,dy \right) dx$$

$$= \int_0^2 \left(x \frac{y^3}{3} \Big|_{y=1}^{y=2} \right) dx = \int_0^2 \left[\frac{8x}{3} - \frac{x}{3} \right] dx = \frac{7}{3} \int_0^2 x\,dx$$

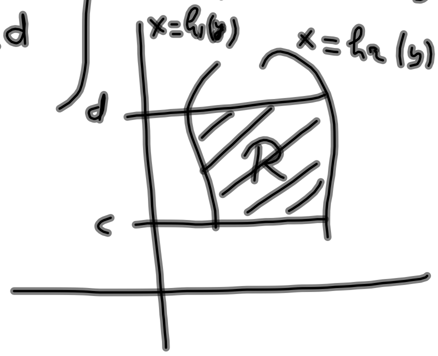
$$= \frac{7}{3} \cdot \frac{x^2}{2} \Big|_0^2 = \frac{14}{3}.$$

<u>Case 2</u> If $R = \{(x,y) : a \leq x \leq b \quad g_1(x) \leq y \leq g_2(x)\}$

then $\displaystyle\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

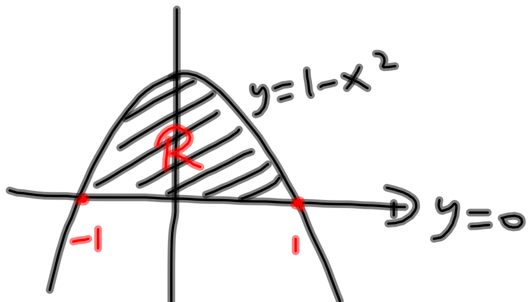
<u>Case 3</u> If $R = \left\{(x,y) \begin{array}{l} h_1(y) \leq x \leq h_2(y) \\ c \leq y \leq d \end{array}\right\}$

then $\displaystyle\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$



<u>Exp:</u> Evaluate the integral

$I = \displaystyle\iint_R (x+2y)\,dA$, where R is the region bounded by $y=0$ and $y = 1-x^2$.



$R: \left\{\begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq y \leq 1-x^2 \end{array}\right\}$

$$I = \int_{-1}^{1}\int_0^{1-x^2} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} xy + y^2 \Big|_{y=0}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \left[x(1-x^2) + (1-x^2)^2\right] dx$$

$$= \int_{-1}^{1} (x - x^3 + 1 + x^4 - 2x^2)\,dx$$

$$= \frac{x^2}{2} - \frac{x^4}{4} + x + \frac{x^5}{5} - \frac{2}{3}x^3 \Big|_{-1}^{1} = 2\left(1 + \tfrac{1}{5} - \tfrac{2}{3}\right) = \frac{16}{15}$$

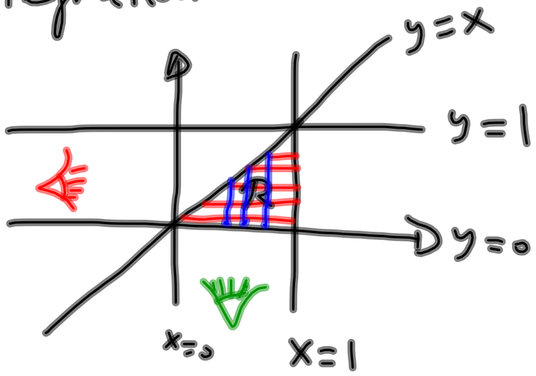
Changing the order of a double integral:

Example: Evaluate the iterated integral:

$$I = \int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$

Remark: We cannot compute this integral with the given order. We should change the order of integration.

$$R = \left\{ \begin{array}{l} y \le x \le 1 \\ 0 \le y \le 1 \end{array} \right\}$$



$$R = \left\{ \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le x \end{array} \right.$$

$$I = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 e^{x^2} \cdot y \Big|_0^x dx$$

$$= \int_0^1 e^{x^2} x\,dx$$

$$= \frac{1}{2} \int_0^1 e^{x^2} \cdot 2x\,dx$$

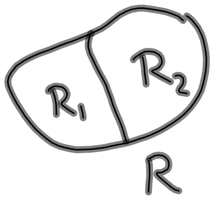
$$= \frac{1}{2} e^{x^2} \Big|_0^1$$

$$= \frac{e-1}{2}$$

$u = x^2$
$du = 2x\,dx$
$\int e^u du = e^u$

Properties $\boxed{1}$ If $R = R_1 \cup R_2$, where R_1 and R_2 don't overlap, then:

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

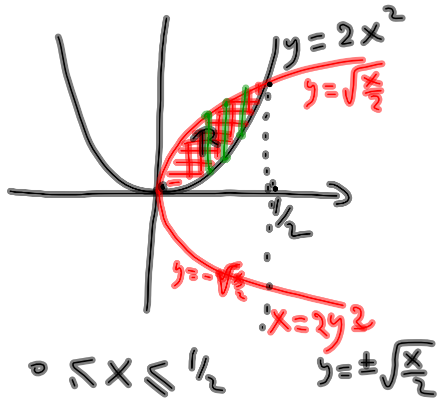


$\boxed{2}$ The area of the region R is:

$$A(R) = \iint_R dA$$

Exp 1 Find the area of the region bounded by the curves $y = 2x^2$ and $x = 2y^2$

Intersections: $\begin{cases} y = 2x^2 & (1) \\ x = 2y^2 & (2) \end{cases}$



(1) into (2): $x = 2\left[2x^2\right]^2 = 8x^4$

$8x^4 - x = 0$

$x(8x^3 - 1) = 0$

$\underline{x = 0}$ $\quad 8x^3 = 1$

$x^3 = \frac{1}{8} \Rightarrow \underline{x = 1/2}$

$R: \begin{cases} 0 \leqslant x \leqslant 1/2 \\ 2x^2 \leqslant y \leqslant \sqrt{\frac{x}{2}} \end{cases}$

$$A(R) = \int_0^{1/2} \int_{2x^2}^{\frac{\sqrt{x}}{\sqrt{2}}} 1\,dy\,dx = \int_0^{1/2} y \Big|_{2x^2}^{\sqrt{\frac{x}{2}}} dx$$

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2}$$

$$= \int_0^{1/2} \left(\frac{\sqrt{x}}{\sqrt{2}} - 2x^2\right) dx = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} x^{3/2} - \frac{2x^3}{3} \Big|_0^{1/2}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}\sqrt{2}} - \frac{2}{3} \cdot \frac{1}{8}$$

$$= 1/6 - \frac{1}{12}$$

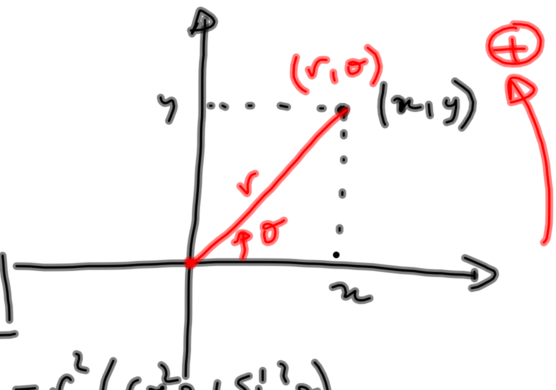
$$= 1/12.$$

Double integrals in polar coordinates:

$\cos\theta = \frac{x}{r}$

$\Rightarrow \boxed{x = r\cos\theta}$

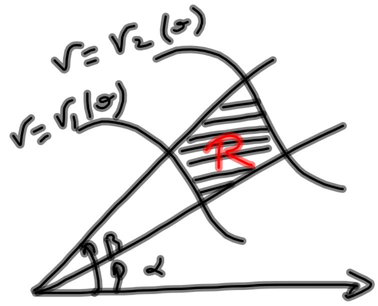
$\sin\theta = \frac{y}{r} \Rightarrow \boxed{y = r\sin\theta}$



$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta)$

$\boxed{x^2+y^2=r^2}$

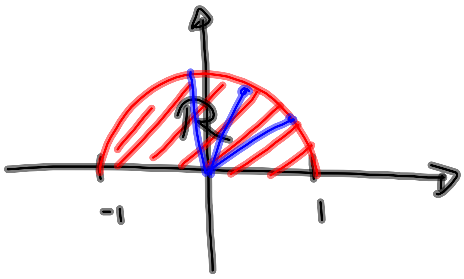
If $R = \left\{ (r,\theta) : \begin{array}{l} \alpha \le \theta \le \beta \\ r_1(\theta) \le r \le r_2(\theta) \end{array} \right\}$,



then $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

Exp: Evaluate $I = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

$R: \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$ $\left(\begin{array}{c} y = \sqrt{1-x^2} \\ y^2 = 1-x^2 \\ x^2+y^2 = 1 \end{array} \right)$



$R: \begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le 1 \end{cases}$

$I = \int_0^\pi \int_0^1 (r^2)^{3/2}\, r\, dr\, d\theta = \int_0^\pi \int_0^1 r^4\, dr\, d\theta$

$= \int_0^\pi \frac{r^5}{5}\Big|_0^1 d\theta = \frac{1}{5}\int_0^\pi d\theta = \pi/5.$