Question	Q1	Q2	Q3	Q4	Q5	Total
Grade						

Q1. **(a)** Without using truth tables, show that $[(p \wedge \neg q) \vee \neg p] \rightarrow r$ is logically equivalent to $\neg r \rightarrow (p \wedge q)$. (3)

$$\begin{aligned}
[(p \wedge \neg q) \vee \neg p] \rightarrow r &\equiv \neg [(p \wedge \neg q) \vee \neg p] \vee r \\
&\equiv [\neg (p \wedge \neg q) \wedge \neg (\neg p)] \vee r \\
&\equiv [(\neg p \vee q) \wedge p] \vee r \\
&\equiv [p \wedge (\neg p \vee q)] \vee r \\
&\equiv (p \wedge \neg p) \vee (p \wedge q)] \vee r \\
&\equiv [F \vee (p \wedge q)] \vee r \\
&\equiv (p \wedge q) \vee r \\
&\equiv r \vee (p \wedge q) \\
&\equiv \neg (\neg r) \vee (p \wedge q) \\
&\equiv \neg r \rightarrow (p \wedge q).
\end{aligned}$$

(b) Use induction to prove the following for every $n \geq 1$:

$$P(n): \frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \cdots + \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}. \quad (4)$$

Basis Step: $P(1)$: LHS $= \frac{2(1)+1}{1^2(1+1)^2} = \frac{3}{1^2 \times 2^2} = \frac{3}{4}$

RHS $= 1 - \frac{1}{(1+1)^2} = 1 - \frac{1}{4} = \frac{3}{4}$ $\therefore$ $P(1)$ is true

(IH)

Induction hypothesis: Assume that $P(k)$ is true for some $k \geq 1$

i.e $\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \cdots + \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(k+1)^2}$ is true

Need to show P(k+1) is true

i.e.

$$\frac{3}{1^2\times 2^2}+\frac{5}{2^2\times 3^2}+\cdots+\frac{2k+1}{k^2\times(k+1)^2}+\frac{\overbrace{2(k+1)+1}^{2k+3}}{(k+1)^2(k+2)^2}=1-\frac{1}{(k+2)^2}$$

From (I.H.)

$$\frac{3}{1^2\times 2^2}+\frac{5}{2^2\times 3^2}+\cdots+\frac{2k+1}{k^2(k+1)^2}=1-\frac{1}{(k+1)^2}$$

$$\Rightarrow \frac{3}{1^2\times 2^2}+\frac{5}{2^2\times 3^2}+\cdots+\frac{2k+1}{k^2(k+1)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}=1-\frac{1}{(k+1)^2}+\frac{2k+3}{(k+1)^2(k+2)^2}$$

$$=1-\left[\frac{1}{(k+1)^2}-\frac{2k+3}{(k+1)^2(k+2)^2}\right]$$

$$=1-\left[\frac{(k+2)^2-(2k+3)}{(k+1)^2(k+2)^2}\right]$$

$$=1-\left[\frac{(k+1)^2}{(k+1)^2(k+2)^2}\right]$$

$$=1-\frac{1}{(k+2)^2}$$

$$(k+2)^2-(2k+3)$$
$$=k^2+4k+4-2k-3$$
$$=k^2+2k+1$$
$$=(k+1)^2$$

∴ By MI the statment is true $\forall n\geq 1$

(c) Let a, b and c be real numbers. Use contraposition to show that if $2a - c > 7$, then $a - b > 1$ or $c - 2b < -5$. (2)

Assume that $a - b \leq 1$ and $c - 2b \geq -5$

then $2a - 2b \leq 2$ and $-c + 2b \leq 5$

so $(2a - 2b) + (-c + 2b) \leq 7$

i.e $2a - c \leq 7$

Q2. **(a)** Let R be the relation on $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ defined by:

$$xRy \Leftrightarrow x - y = 10.$$

Determine whether R is reflexive, symmetric, antisymmetric or transitive. (4)

R is not reflexive as for $0 \in \mathbb{Z}^+$, $0 - 0 = 0 \neq 10$

R is not symmetric for $20 - 10 = 10$ but $10 - 20 \neq 10$

R is not transitive as $20 R 10$, $30 R 20$ but $30 \not R 10$ ($20 - 10 = 10$, $30 - 20 = 10$ but $30 - 10 = 20$)

R is antisymmetric ($\forall\, xRy \wedge yRx \rightarrow x = y$)

since $xRy \wedge yRx$ is F

then R is antisymmetric.

(b) Let E be the relation on $\mathbb{Z}$ defined by $mEn \Leftrightarrow 4 \mid (m^2 - n^2)$.

(i) Prove that E is an equivalence relation. (3)

* E is reflexive: as $\forall m \in \mathbb{Z}$, $4 \mid m^2 - m^2$ (as $4 \mid 0$)
* E is symmetric for if mEn then $4 \mid m^2 - n^2$

$\Rightarrow m^2 - n^2 = 4k$ ($\exists k \in \mathbb{Z}$)

$\Rightarrow n^2 - m^2 = 4(-k)$ ($-k \in \mathbb{Z}$)

$\therefore nEm$

* E is transitive if $4 \mid m^2 - n^2$ and $4 \mid n^2 - r^2$

$\Rightarrow m^2 - n^2 = 4k$ and $n^2 - r^2 = 4l$ $k, l \in \mathbb{Z}$

$\Rightarrow m^2 - n^2 + n^2 - r^2 = 4(k + l)$

i.e. $m^2 - r^2 = 4(k + l)$ $k + l \in \mathbb{Z}$

$\therefore 4 \mid m^2 - r^2$

(ii) Show that $[m] = [7m]$ for every $m \in \mathbb{Z}$. (1)

as $7mEm$ ($(7m)^2 - m^2 = 49m^2 - m^2 = 48m^2 = 4(12m^2)$)

$\therefore [m] = [7m]$

(iii) Show that $m + 1 \notin [m]$ for every $m \in \mathbb{Z}$. (1)

as $m+1 \not E m$ ($(m+1)^2 - m^2 = m^2 + 2m + 1 - m^2 = 2m + 1 \neq 4k$ $\forall k \in \mathbb{Z}$)

Q3. **(a)** Let G be a simple graph with degree sequence $x, x, x, x, x, x, 2x, 4x$ and having 6 edges.

(i) Find x. (1)

From HST $\sum \deg(v) = 2(6)$

$12x = 12$ $\therefore x = 1$

(ii) Show that G cannot be connected. (1)

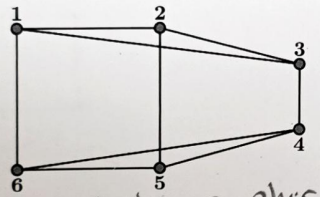
Since G has 8 vertices and 6 edges
(For G to be connected it must have at least 7 edges)
Or 1 vertix is not connected to any other vertix. (isolated)

(iii) Find the number of edges of the complement $\overline{G}$ of G. (1)

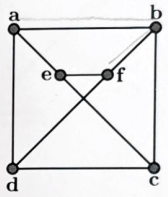
$$E_{\overline{G}} + E_G = \frac{8(8-1)}{2}$$

$$E_{\overline{G}} = 28 - 6 = 22$$

(b) Determine if the following graphs H and I are isomorphic. (2)



H



I

They are not isomorphic
The Graph H contains simple circuits of length 3 but I doesn't have,
(both graphs has same number of vertices and edges) and same degree
(or I is bipartite whereas H is not)
details are in(c))

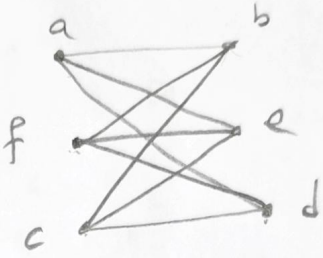
(c) Determine whether the graph I in **(b)** is bipartite, and if so, then find a bipartite representation. (2)

yes it is

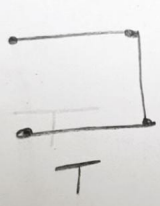
the bipartite representation is

$V_1 = \{a, c, f\}$, and

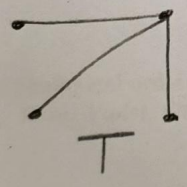
$V_2 = \{b, d, e\}$



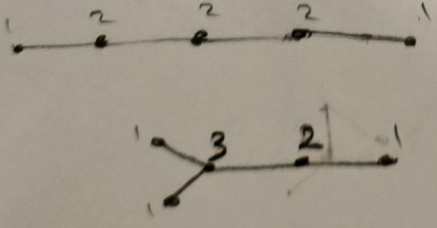
Q4.(a) (i) Give an example of a tree T, whose complement $\overline{T}$ is a tree. (1)



(ii) Give an example of a tree T, whose complement $\overline{T}$ is not a tree. (1)

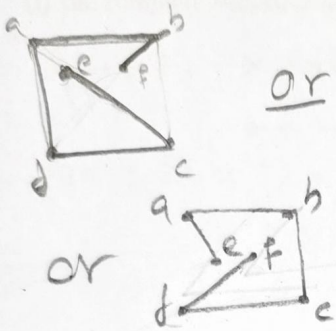


(iii) Draw 3 (nonisomorphic) trees with 5 vertices. (2)



(b) For the graph I in **Q3(b)**, find a spanning tree with root f,

(i) using *depth-first* search; (1)



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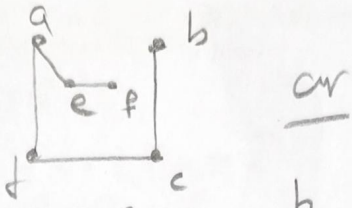

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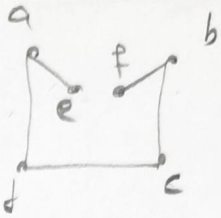

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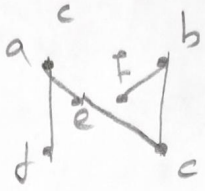
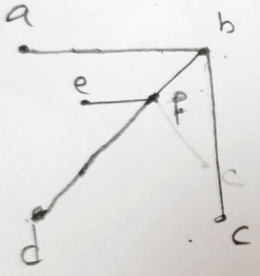
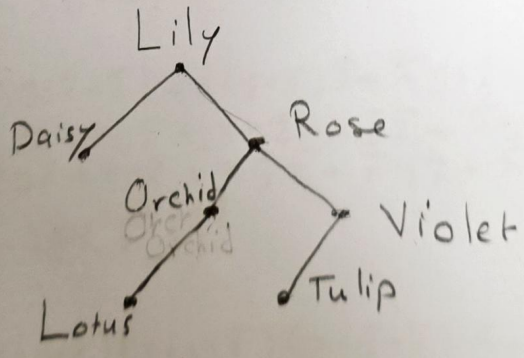


(ii) using *breadth-first* search. (1)



(c) Using alphabetical order, form a binary search tree for the words:
Lily, Daisy, Rose, Violet, Orchid, Tulip, Lotus. (2)



Another method please see the back of this page →

Q5. **(a)** For the Boolean function $f(x, y, z) = (x + y\bar{z})(\bar{x} + y)$, find
(i) the complete sum-of-products expansion (CSP); (2)

$$f(x,y,z) = (x + y\bar{z})(\bar{x} + y)$$
$$= x\bar{x} + xy + \bar{x}y\bar{z} + y\bar{z}$$
$$\therefore f(x,y,z) = xy + \bar{x}y\bar{z} + y\bar{z}$$
$$= xy(z+\bar{z}) + \bar{x}y\bar{z} + (x+\bar{x})y\bar{z}$$
$$= xyz + xy\bar{z} + \bar{x}y\bar{z} + xy\bar{z} + \bar{x}y\bar{z}$$
$$= xyz + xy\bar{z} + \bar{x}y\bar{z}$$

(ii) the complete product-of-sums expansion (CPS). (2)

$$CPS(f) = [CSP(f^d)]^d$$
$$f^d(x,y,z) = x(y+\bar{z}) + \bar{x}y = xy + x\bar{z} + \bar{x}y$$
$$CSP(f^d) = xy(z+\bar{z}) + x(y+\bar{y})\bar{z} + \bar{x}y(z+\bar{z})$$
$$= xyz + xy\bar{z} + xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z}$$
$$= xyz + xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}y\bar{z}$$
$$CPS(f) = [CSP(f^d)]^d$$
$$= (x+y+z)(x+y+\bar{z})(x+\bar{y}+\bar{z})(\bar{x}+y+z)(\bar{x}+y+\bar{z})$$

x	y	z	$x+y\bar{z}$	$\bar{x}+y$	$f(x,y,z)$
1	1	1	1	1	1
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	0	0
0	1	1	0	1	0
0	1	0	1	1	1
0	0	1	0	1	0
0	0	0	0	1	0

From the table we conclude that

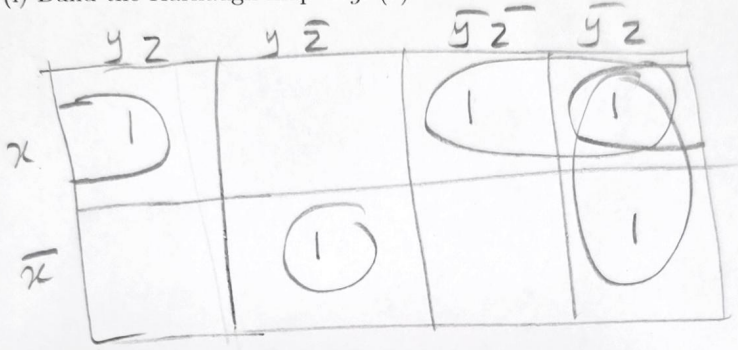
$$CSp(f) = xyz + xy\bar{z} + \bar{x}y\bar{z}$$

and

$$CPS(f) = (\bar{x}+y+\bar{z})(\bar{x}+y+z)(x+\bar{y}+\bar{z})(x+y+\bar{z})(x+y+z)$$

(b) Let $g(x, y, z) = xyz + x\bar{y}\,\bar{z} + x\bar{y}z + \bar{x}\,y\bar{z} + \bar{x}\,\bar{y}z$ be a Boolean function.

(i) Build the Karnaugh map of g. (1)



(ii) Simplify g (i.e., write it in MSP form). (2)

$g(x, y, z) = xz + x\bar{y} + \bar{y}z + \bar{x}y\bar{z}$