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Question I: (8points)

Question	1	2	3	4	5	6	7	8
Answer	b	c	c	a	b	a	b	a

Choose the correct answer, then fill in the table above:

(1) If $p \to q$ is false, then $\neg p \vee q$ is

(a) True.
(b) False.

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(2) The proposition " $x^2 \leq x$ *whenever* $0 \leq x \leq 1$" is equivalent to:

(a) $x^2 \leq x$ *implies* $0 \leq x \leq 1$.
(b) $x^2 \leq x$ *if and only if* $0 \leq x \leq 1$.
(c) If $0 \leq x \leq 1$ then $x^2 \leq x$.
(d) None of the previous.

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(3) The proposition $\neg \exists x: (x^3 = 1 \wedge 2x = 1)$ is equivalent to:
(a) $\exists x: (x^3 \neq 1 \vee 2x \neq 1)$.
(b) $\forall x: (x^3 \neq 1 \wedge 2x \neq 1)$.
(c) $\forall x: (x^3 \neq 1 \vee 2x \neq 1)$.
(d) None of the previous.

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(4) The proposition "$\neg[p \vee \neg p] \to q$" is a

$F \to q$

(a) Tautology.
(b) Contradiction.
(c) Contingency.

(5) The truth value of the statement $\exists! x \in \mathbb{R}$ such that $x^2 - 5 = 0$ is:

(a) True.

(b) False.

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(6) The inverse of the proposition $if\ x + y = 2\ then\ 2x + 2y > 2$" is:

(a) if $x + y \neq 2\ then\ 2x + 2y \leq 2$.

(b) if $2x + 2y > 2\ then\ x + y = 2$.

(c) if $2x + 2y \leq 2\ then\ x + y \neq 2$.

(d) None of the previous.

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(7) If the domain of the propositional function $P(x)$ consists of the integers $1, 2, 3$ *and* 4, then the following is equivalent to the statement $\neg\forall x P(x)$:

(a) $\neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$.

(b) $\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$.

(c) $P(1) \vee P(2) \vee P(3) \vee P(4)$.

(d) None of the previous.

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(8) Let $P(x, y)$ be the statement " $y - 2x$ *is odd* " then the following is true:

(a) $P(1,5)$.

(b) $P(1,2)$.

(c) $P(3,2)$.

(d) None of the [illegible]ous

Question II: (2+2+2=6 points)

A. **Without using truth tables** prove the following:

$$\neg p \to (\neg p \wedge q) \equiv \neg p \to q.$$

$\neg p \to (\neg p \wedge q) \equiv \neg(\neg p) \vee (\neg p \wedge q)$ as $p \to q \equiv \neg p \vee q$

$\equiv p \vee (\neg p \wedge q)$ Double negation

$\equiv (p \vee \neg p) \wedge (p \vee q)$ Distributive law.

$\equiv T \wedge (p \vee q)$, as $p \vee \neg p \equiv T$

$\equiv p \vee q$, commutative law + $p \wedge T$

$\equiv \neg p \to q$ as $\cancel{p \to q} \equiv \neg p \vee q$.

$\therefore \neg p \to (\neg p \wedge q) \equiv \neg p \to q$.

B. **Prove that: $2^n + 1 \geq n^2$**, for all nonnegative integers n less than 4.

Domain $= \{0, 1, 2, 3\}$.

$P(n)$: $2^n + 1 \geq n^2$.

$P(0)$: $2^0 + 1 = 2$, $0^2 = 0$ $\therefore 2 \geq 0$ is true

$P(1)$: $2^1 + 1 = 3$, $1^2 = 1$ $\therefore 3 \geq 1$ is true.

$P(2)$: $2^2 + 1 = 5$, $2^2 = 4$, $\therefore 5 \geq 4$ is true

$P(3)$: $2^3 + 1 = 9$, $3^2 = 9$ $\therefore 9 \geq 9$ is true.

$\therefore P(n)$ is true $\forall n \in \{0, 1, 2, 3\}$.

C. Prove by **contradiction**: "*If* $m \neq 0$ *is rational and n is irrational then mn is irrational*".

p: $m \neq 0$ is rational and n is irrational

q: mn is irrational.

Proof by contradiction: Suppose that $p \wedge \neg q$ is true.

That is $0 \neq m$ rational and n irrational, and mn is rational.

Now $\exists\, p, q \in \mathbb{Z}$ s.t $m = \frac{p}{q}$, $q \neq 0$ and as $m \neq 0$, $p \neq 0$.

As mn is rational, then $\exists\, a, b \in \mathbb{Z}$, $b \neq 0$ s.t

$$mn = \frac{a}{b}$$

as $m \neq 0$

$$n = \frac{a}{b} \cdot \frac{1}{m}$$

$$= \frac{a}{b} \cdot \frac{q}{p} = \frac{aq}{bp}$$, $aq \in \mathbb{Z}$ $bp \in \mathbb{Z}$, $bp \neq 0$ as $b \neq 0 \wedge p \neq 0$.

$\therefore$ n is rational. A contradiction with n being irrational. Thus $p \rightarrow q$ is true.

$\therefore$ If $0 \neq m$ is rational and n irrational then mn irrational

Question III: (1+5=6 points)

A. Prove that the statement "*for every* $n \in \mathbb{Z}$, $n^2 > 2n$" is false.

The statement $\forall\, n \in \mathbb{Z}$ $n^2 > 2n$ is false as

$n = 0$ is a counter example:

$$0^2 \not> 2(0).$$

$$0 \not> 0.$$

B. For every integer n, prove that: " n is even if and only if $3n^2 + 5$ is odd"

$p \leftrightarrow q$

p: n even, q: $3n^2+5$ odd.

(1) $p \to q$: n even $\to 3n^2+5$ is odd.

Suppose n is even, $\exists k \in \mathbb{Z}$ s.t $n = 2k$

$n^2 = 4k^2$

$3n^2+5 = 12k^2+5$

$= 12k^2+4+1$

$= 2(6k^2+2)+1$, put $6k^2+2 = m$ for some $m \in \mathbb{Z}$

$= 2m+1$,

$\therefore 3n^2+5$ is odd.

We proven n even $\to 3n^2+5$ odd.

(2) $q \to p$: $3n^2+5$ odd $\overset{?}{\to}$ n even.

Proving by contraposition: $\neg p \to \neg q$: n odd $\overset{?}{\to} 3n^2+5$ even

Suppose n is odd, $\exists k \in \mathbb{Z}$ s.t $n = 2k+1$

$n^2 = 4k^2+4k+1$

$3n^2+5 = 12k^2+12k+3+5$

$= 12k^2+12k+8$

$= 2(6k^2+6k+4)$. put $6k^2+6k+4 = m$ for some $m \in \mathbb{Z}$

$= 2m$

$\therefore 3n^2+5 = 2m$.

We proved n odd $\to 3n^2+5$ even, that is equivalent $3n^2+5$ odd $\to$ n even.

From (1), (2) we proven $p \leftrightarrow q$.

defined inductively as:

$$a_0 = 0, \quad a_1 = 1, \; a_n = 3a_{n-1} - 2a_{n-2}, \; \forall n \geq 2.$$

Using **Strong Induction** prove that:

$$\boldsymbol{a_n = 2^n - 1}, \; \textit{for all nonnegative integers } n.$$

$P(n)$: $a_n = 2^n - 1$.

(1) Basis Step: (1) $n=0$: $a_0 = 0$ (given), $2^0 - 1 = 0$.

$\therefore$ R.H.S = L.H.S $\therefore$ true $n=0$

(2) $n=1$: $a_1 = 1$ given, R.H.S $= 2^1 - 1 = 1$

$\therefore$ L.H.S = R.H.S $\therefore$ True for $n=1$.

(2) Inductive Step: $P(0) \land P(1) \land P(2) \land \dots \land P(k) \overset{??}{\rightarrow} P(k+1)$

Suppose for some $k \in \mathbb{Z}$ that $P(j)$ is true, $j \leq k$.

$a_j = 2^j - 1$, (*) I.H

We want to prove that $P(k+1)$ is true:

$a_{k+1} \overset{??}{=} 2^{k+1} - 1$

Now $a_{k+1} = 3a_k - 2a_{k-1}$ from $a_n = 3a_{n-1} - 2a_{n-2}$.

$= 3(2^k - 1) - 2(2^{k-1} - 1)$

$= 3(2^k - 1) - 2^k + 2$

$= 3 \cdot 2^k - 3 - 2^k + 2$

$= 2^k[3-1] ~~+~~ - 1$

$= 2^{k+1} - 1 =$ R.H.S

$\therefore P(k+1)$ is true. We prove $P(0) \land P(1) \land \dots \land P(k) \rightarrow P(k+1)$ is true. From pricipal of strong induction we proved $P(n)$ is true $\forall$ nonegitive integer.

Good Luck ☺