First semester	Second exam	King Saud University
(5^{th} Nov. 2025)-(14/5/1447)	131 Math	College of Science
Without calculators	Time: 90 minutes	Mathematics Department

Name: **University Number:**

Q_1: Prove or disprove the following:

1- Let $x, y \in \mathbb{Z}$. Then x and y are of the same parity if and only if $x + y$ is even. (4 marks)

2- Let $n \in \mathbb{Z}$. If $5 \nmid (n^2 + 4)$, then $5 \nmid (n - 1)$ and $5 \nmid (n + 1)$. (4 marks)

3- For sets A, B and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$. (4 marks)

4- There is no smallest positive real number. (3 marks)

5- For every odd positive integer n, $3 \mid (n^2 - 1)$. (2 marks)

6- If 3 is an even number, then $3^2=9$. (1 mark)

7- A sequence $\{a_n\}$ is defined recursively by

$$a_1 = 1,\ a_2 = 3 \text{ and } a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Then $a_n = 2n - 1$ for all $n \in \mathbb{N}$. (4 marks)

8- For each positive integer n, let $P(n)$ be a statement. If

(1) $P(1)$ is true and

(2) the implication: $P(k) \Rightarrow P(k + 1)$, is true for every positive integer k,

then $P(n)$ is true for every positive integer n. (3 marks)

Answers

Q_1:

1- First, assume that x and y are of the same parity. We consider two cases.
Case 1. *x and y are even.* Then $x = 2a$ and $y = 2b$ for some integers a and b. So, $x + y = 2a + 2b = 2(a + b)$. Since $a + b \in \mathbb{Z}$, the integer $x + y$ is even.
Case 2. *x and y are odd.* Then $x = 2a + 1$ and $y = 2b + 1$, where $a, b \in \mathbb{Z}$. Therefore, $x + y = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1)$. Since $a + b + 1$ is an integer, $x + y$ is even.
For the converse, assume that x and y are of opposite parity. Without loss of generality, assume that x is even and y is odd. Then $x = 2a$ and $y = 2b + 1$, where $a, b \in \mathbb{Z}$. Then $x + y = 2a + (2b + 1) = 2(a + b) + 1$. Since $a + b \in \mathbb{Z}$, the integer $x + y$ is odd.

2- We will use the contrapositive proof.
Assume that $5 \mid (n - 1)$ or $5 \mid (n + 1)$. We consider these two cases.
Case 1. $5 \mid (n - 1)$. Then $n - 1 = 5a$ for some integer a. So, $n = 5a + 1$. Hence, $n^2 + 4 = (5a + 1)^2 + 4 = (25a^2 + 10a + 1) + 4 = 5(5a^2 + 2a + 1)$. Since $5a^2 + 2a + 1 \in \mathbb{Z}$, it follows that $5 \mid (n^2 + 4)$.
Case 2. $5 \mid (n + 1)$. Then $n + 1 = 5b$, where $b \in \mathbb{Z}$, and so $n = 5b - 1$. Hence, $n^2 + 4 = (5b - 1)^2 + 4 = (25b^2 - 10b + 1) + 4 = 5(5b^2 - 2b + 1)$. Since $5b^2 - 2b + 1 \in \mathbb{Z}$, it follows that $5 \mid (n^2 + 4)$.

3- We first show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.
Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$.

Thus, $y \in B$ or $y \in C$, say the former. Then $(x, y) \in A \times B$ and so $(x, y) \in (A \times B) \cup (A \times C)$. Consequently, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Next, we show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$, say the former. Then $x \in A$ and $y \in B \subseteq B \cup C$. Hence, $(x, y) \in A \times (B \cup C)$, implying that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

4- Assume, to the contrary, that there is a smallest positive real number, say r. Since $0 < r/2 < r$, it follows that $r/2$ is a positive real number that is smaller than r. This, however, is a contradiction.

5- Since $3 \nmid 8$, so $3 \nmid (3^2 - 1)$. It follows that $n = 3$ is a counterexample.

6- Since 3 is an odd number, so (3 is even) is a false statement and the implication is true.

7- We proceed by induction. Since $a_1 = 2 \cdot 1 - 1 = 1$ and $a_2 = 2 \cdot 2 - 1 = 3$, the formula holds for $n = 1$ and $n = 2$. Assume for an arbitrary positive integer k that $a_i = 2i - 1$ for all integers i with $1 \leq i \leq k$. We show that $a_{k+1} = 2(k + 1) - 1 = 2k + 1$. Since $a_2 = 3$, it follows that $a_{k+1} = 2k + 1$ when $k = 1$. Hence, we may assume that $k \geq 2$. Since $k + 1 \geq 3$, it follows that $a_{k+1} = 2a_k - a_{k-1} = 2(2k - 1) - (2k - 3) = 2k + 1$, which is the desired result. By the Strong Principle of Mathematical Induction, $a_n = 2n - 1$ for all $n \in \mathbb{N}$.

8- Assume, to the contrary, that the theorem is false. Then conditions (1) and (2) are satisfied but there exist some positive integers n for which $P(n)$ is a false statement. Let $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. Since S is a nonempty subset of $\mathbb{N}$, it follows by the Well-Ordering Principle that S contains a least element s. Since $P(1)$ is true, $1 \notin S$. Thus, $s \geq 2$ and

$s-1 \in \mathbb{N}$. Therefore, $s-1 \notin S$ and so $P(s-1)$ is a true statement. By condition (2), $P(s)$ is also true and so $s \notin S$. This, however, contradicts our assumption that $s \in S$.