First semester	Final exam	King Saud University
(21st Dec. 2025)-(1/7/1447)	131 Math	College of Science
Without calculators	Time: 3 hours	Mathematics Department

Q₁: (8 marks)

(i) Show that $P \Rightarrow Q \equiv (\sim P) \vee Q$ (2 marks)

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(ii) $\sim(\forall \epsilon > 0, \exists \delta > 0, \forall x \in D, |x-1| < \delta \Rightarrow |f(x)-3| < \epsilon)$ (2 marks)

$$\equiv \exists \epsilon > 0, \forall \delta > 0, \exists x \in D, \sim(|x-1| < \delta \Rightarrow |f(x)-3| < \epsilon)$$

$$\equiv \exists \epsilon > 0, \forall \delta > 0, \exists x \in D, |x-1| < \delta \wedge |f(x)-3| \geq \epsilon$$

(iii) If $A, B \subseteq \Omega$ such that $|\Omega|=10$, $|A|=5$, $|B|=4$ and $|A \cup B|=8$, then $|A \cap B|=1$, $|\overline{A \cap B}|=9$, $|A-B|=4$ and $|P(B)|=2^4=16$ (2 marks)

(iv) Give an example of a relation on A={1,2,3}that is symmetric and transitive but NOT reflexive. (1 mark)

R={(1,1)}.

(v) Give an example of two sets, the first one is countably infinite and the second one is uncountable. (1 mark)

ℕ is countably infinite (denumerable)

ℝ is uncountable.

Q₂: (11 Marks)

Answer the following questions:

1- Show that $5n^2+3n+1$ is odd for all integer n. (3 marks)

We have the following two theorems for all integers x and y:

"x and y are of the same parity if and only if $x + y$ is even".

"xy is even if and only if x is even or y is even".

Case 1: If n is even, then $5n^2$ and $3n$ are even using the second theorem. Thus, $5n^2+3n$ is even using the first theorem. Since 1 is odd, the first theorem implies $5n^2+3n+1$ is odd.

Case 2: If n is odd, then n^2 is odd and hence $5n^2$ and $3n$ are odd using the second theorem. Thus, $5n^2+3n$ is even using the first theorem. Since 1 is odd, the first theorem implies $5n^2+3n+1$ is odd.

2- Evaluate the proposed proof of the following result: "If x and y are even numbers, then $x + y$ is even".

Proof: Take the even numbers $x = 2k$ and $y = 2k$ for some integer k. Observe that $x + y = 2k + 2k = 4k = 2(2k)$. Since $2k$ is an integer, so $x + y$ is even. (2 marks)

The proof is false since it took $x = y = 2k$ while it should be taken arbitrary even numbers (not necessary x equals y).

3- Define a relation R on $\mathbb{Z}$ as follows: xRy if and only if $x + y$ is even. Show that R is an equivalence relation and find [1]. (4 marks)

We need the following theorem for all integers x and y:

"x and y are of the same parity if and only if $x + y$ is even".

Since x and x have the same parity, $x + x$ is even and hence xRx for all integers x and R is reflexive.

If xRy, $x + y$ is even. But the addition is commutative on $\mathbb{Z}$. So, $y + x$ is even. Therefore, yRx and R is symmetric.

Now, if xRy and yRz, then $x + y$ and $y + z$ are even. Using the above theorem, x and y are of the same parity, and y and z are of the same parity. Thus, x and z are of the same parity and hence $x + z$ is even. Therefore, xRz and R is transitive.

Since R is reflexive, symmetric and transitive, it is an equivalence relation and using the above theorem, we have that

$$[1] = \{x \in \mathbb{Z} | xR1\} = \{x \in \mathbb{Z} | x + 1 \text{ is even}\} = \{x \in \mathbb{Z} | x \text{ is odd}\}$$

4- Is $P = \{(-\infty, -1), [-1,1], (1, \infty)\}$ a partition of the set of real numbers? Explain your answer. (2 marks)

Yes, since each subset is nonempty, the intersection of each two different subsets is empty and the union of the three subsets is $\mathbb{R}$.

Q_3: (6 marks)

Let $f: \mathbb{R} \to \mathbb{R}$ such that $f(x)=2x$.

(i) Show that f is a function. (2 marks)

If $x, y \in \mathbb{R}$ such that $x = y$, then $2x = 2y$ and hence $f(x) = f(y)$. So, f is a function.

(ii) Show that f^{-1} is a function. (3 marks)

We have the following theorem:

"Let f: A → B be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. Furthermore, if f is bijective, then f^{-1} is also bijective".

So, we need to show that f is bijective.

If $x, y \in \mathbb{R}$ such that $f(\mathrm{y}) = \mathrm{f(x)}$, then $2\mathrm{y} = 2\mathrm{x}$ which implies that $\mathrm{y} = \mathrm{x}$ and f is one-to-one. Now, for all $y \in \mathbb{R}$ take $x = \frac{1}{2}y$. Then $x \in \mathbb{R}$ and $\mathrm{f(x)} = f\left(\frac{1}{2}y\right) = 2\left(\frac{1}{2}y\right) = y$ and f is onto.

Since f is 1-1 and onto, it is bijective and hence f^{-1} is a function, using the above theorem.

(iii) Find $f(\{1,2\})$ and $f^{-1}(0)$. (1 mark)

$f(1) = 2, f(2) = 4$ and hence $f(\{1,2\}) = \{2,4\}$.

$y = f(x) = 2x$ implies $f^{-1}(y) = x = \frac{1}{2}y$. So, $f^{-1}(0) = 0$.

Q$_4$: (5 marks)

Prove the following, using the formal definition of the limit (ϵ, δ or ϵ, N)

(i) $\lim_{n \to \infty} \frac{1}{n} = 0$ (2 marks)

We need to show that:

$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n > N \Longrightarrow |\frac{1}{n} - 0| < \epsilon$. Let $\epsilon > 0$ be given. Observe that $\left|\frac{1}{n} - 0\right| < \epsilon \Longrightarrow \frac{1}{n} < \epsilon$. So, $\frac{1}{\epsilon} < n$. So, choose $N = \lceil \frac{1}{\epsilon} \rceil$. Then if $n \in \mathbb{N}$ and $n > N$, then $|\frac{1}{n} - 0| < \epsilon$. Hence, $\lim_{n \to \infty} \frac{1}{n} = 0$.

(ii) $\lim_{x \to 2} x^2 = 4$. (3 marks)

We need to show that:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, 0 < |x - 2| < \delta \Rightarrow |x^2 - 4| < \epsilon$$

Let $\epsilon > 0$ be given. Observe that $|x^2 - 4| < \epsilon \Longrightarrow |\mathrm{x} - 2||\mathrm{x} + 2| < \epsilon$ (*). Now, suppose that $\delta \le 1$. So, $|\mathrm{x} - 2| < \delta \le 1$ and then $-1 < \mathrm{x} - 2 < 1$. This implies that $1 < \mathrm{x} < 3$ and hence $3 < \mathrm{x} + 2 < 5$. Thus, $|x + 2| < 5$. So, $|\mathrm{x} - 2||\mathrm{x} + 2| < 5|x - 2| < 5\delta$. Using (*), we need $5\delta = \epsilon$ or $\delta = \frac{\epsilon}{5}$. But we have taken $\delta \le 1$ before. Thus, choose $\delta = \min\{1, \frac{\epsilon}{5}\}$ and hence $\delta \le \frac{\epsilon}{5}$. Therefore, $\forall x \in \mathbb{R}$, if $0 < |x - 2| < \delta \le \frac{\epsilon}{5}$, then:

$$|x^2 - 4| = |\mathrm{x} - 2||\mathrm{x} + 2| < \left(\frac{\epsilon}{5}\right)(5) = \epsilon$$

So, we could find a suitable δ for the given (arbitrary) ϵ.

Q$_5$: (10 marks)

(i) For $a, b \in \mathbb{R} - \{-2\}$, define: $a * b = ab + 2a + 2b + 2$, where the operations indicated in $ab + 2a + 2b + 2$ are ordinary addition and multiplication in $\mathbb{R}$. Show that $*$ is a binary operation on $\mathbb{R} - \{-2\}$. (2 marks)

The aim is: if $a, b \in \mathbb{R} - \{-2\}$, then $a * b \in \mathbb{R} - \{-2\}$. Assume, to the contrary, that there exists some pair $x, y \in \mathbb{R} - \{-2\}$ such that $x * y \notin \mathbb{R} - \{-2\}$. Thus, $x * y = xy + 2x + 2y + 2 = -2$. This equation is equivalent to $(x + 2)(y + 2) = 0$, so either $x = -2$ or $y = -2$, which is impossible since $x, y \in \mathbb{R} - \{-2\}$. Hence, $*$ is a binary operation on $\mathbb{R} - \{-2\}$.

(ii) Let $G = \{(x, 0) | x \in \mathbb{R}\}$. For $(x, 0), (y, 0) \in G$, define: $(x, 0) * (y, 0) = (x + y, 0)$ (where $x + y$ is the ordinary addition in $\mathbb{R}$). Show that $(G,*)$ is an abelian group. (4 marks)

1- Since $(x, 0) * (y, 0) = (x + y, 0) \in G$, So $*$ is a binary operation on G.

2- Associativity: For all real numbers x, y and z, and using the definition of $*$ and the associativity of real numbers, we have that:

$$[(x, 0) * (y, 0)] * (z, 0) = (x + y, 0) * (z, 0)$$
$$= ([x + y] + z, 0) = (x + [y + z], 0)$$
$$= (x, 0) * (y + z, 0) = (x, 0) * [(y, 0) * (z, 0)]$$

3- (0,0) is the identity since for a real number x, we have that:

$$(0,0) * (x, 0) = (0 + x, 0) = (x, 0) = (x + 0,0)$$
$$= (x, 0) * (0,0)$$

4- For each $(x, 0) \in G$, $\exists (-x, 0) \in G$ Such that:

$$(x, 0) * (-x, 0) = (x + (-x), 0) = (0,0)$$

In the same way, we can show that $(-x, 0) * (x, 0) = (0,0)$ So, $(-x, 0)$ is the inverse of $(x, 0)$ and $(x, 0)^{-1} = (-x, 0)$.

5- Commutativity: For all real numbers x and y, and using the definition of $*$ and the commutativity of real numbers, we have that:

$$(x, 0) * (y, 0) = (x + y, 0) = (y + x, 0) = (y, 0) * (x, 0)$$

1 to 5 imply that $(G,*)$ is an abelian group.

(iii) Let (R,+,·) be a ring. Show that $a(0)=0$ for all $a \in R$. (2 marks)

Using the property of the additive identity and the right distributive law, observe that

$$a(0) = a(0 + 0) = a(0) + a(0).$$

Adding $-a(0)$ to both sides, we get that

$$0=-a(0)+a(0)= -a(0)+[a(0)+a(0)]$$

$$=[-a(0)+a(0)]+a(0)=0+ a(0)=a(0)$$

Using the additive inverse property and the associativity.

(iv) Prove that the set of even integers E and the set of integers $\mathbb{Z}$ have the same cardinality. (2 marks)

Define $f: \mathbb{Z} \to E$ by $f(x) = 2x$ for all integers x. If $x, y \in \mathbb{Z}$ such that $f(\mathrm{y}) = \mathrm{f(x)}$, then $2\mathrm{y} = 2\mathrm{x}$ which implies that $\mathrm{y} = \mathrm{x}$ and f is one-to-one. Now, If $y \in E$, then $y = 2m$ for some integer m. Take $x = m$. Then $x \in \mathbb{Z}$ and $\mathrm{f(x)} = f(m) = 2m = y$ and f is onto.

Since f is 1-1 and onto, it is bijective and hence E and $\mathbb{Z}$ have the same cardinality.