# Telescoping Sums

## Section 5.2 — Exercises 34, 35 and 36

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**The one idea behind all three.** Suppose the summand can be written as a difference of consecutive values of a single function,

$$a_k = f(k) - f(k+1).$$

Then in the sum every interior term appears twice with opposite signs:

$$\sum_{k=m}^{n} [f(k) - f(k+1)] = \underbrace{[f(m) - f(m+1)]}_{\text{first}} + [f(m+1) - f(m+2)] + \cdots + \underbrace{[f(n) - f(n+1)]}_{\text{last}}.$$

Everything cancels in pairs except the very first and the very last piece, leaving

$$\boxed{\sum_{k=m}^{n} [f(k) - f(k+1)] = f(m) - f(n+1).}$$

Such a sum is called *telescoping*, because it collapses like a folding telescope. The whole task, in each exercise below, is to identify $f$.

## Exercise 34

$$\sum_{k=2}^{20} [\sin(k-1) - \sin k]$$

*Identify the function.*

Take $f(k) = \sin(k-1)$. Then $f(k+1) = \sin k$, so the summand is exactly $f(k) - f(k+1)$.

*Write out enough terms to see the cancellation.*

$$\underbrace{(\sin 1 - \sin 2)}_{k=2} + \underbrace{(\sin 2 - \sin 3)}_{k=3} + \cdots + \underbrace{(\sin 19 - \sin 20)}_{k=20}$$

Each $\sin 2, \sin 3, \ldots, \sin 19$ occurs once positively and once negatively.

*Apply the formula with $m = 2$ and $n = 20$.*

$$\sum_{k=2}^{20} [\sin(k-1) - \sin k] = f(2) - f(21) = \sin 1 - \sin 20$$

$$\boxed{\sum_{k=2}^{20} [\sin(k-1) - \sin k] = \sin 1 - \sin 20 \approx -0.0715}$$

The angles are in radians. Note that the answer is *not* zero: the sum of nineteen sines has collapsed to a difference of two of them.

## Exercise 35

$$\sum_{k=7}^{30}\left(\sqrt{k-4}-\sqrt{k-3}\right)$$

*Identify the function.*

Take $f(k)=\sqrt{k-4}$. Then $f(k+1)=\sqrt{k-3}$, which is the second term.

*Write out enough terms.*

$$\underbrace{\left(\sqrt{3}-\sqrt{4}\right)}_{k=7}+\underbrace{\left(\sqrt{4}-\sqrt{5}\right)}_{k=8}+\cdots+\underbrace{\left(\sqrt{26}-\sqrt{27}\right)}_{k=30}$$

*Apply the formula with $m=7$ and $n=30$.*

$$\sum_{k=7}^{30}\left(\sqrt{k-4}-\sqrt{k-3}\right)=f(7)-f(31)=\sqrt{3}-\sqrt{27}$$

*Simplify.*

Since $\sqrt{27}=\sqrt{9\cdot 3}=3\sqrt{3}$,

$$\sqrt{3}-3\sqrt{3}=-2\sqrt{3}.$$

$$\boxed{\sum_{k=7}^{30}\left(\sqrt{k-4}-\sqrt{k-3}\right)\;=\;-2\sqrt{3}\;\approx\;-3.4641}$$

The sum is negative because each term is negative: $\sqrt{k-4}<\sqrt{k-3}$ for every $k$. That is a useful check before doing any algebra.

## Exercise 36

$$\sum_{k=1}^{40}\frac{1}{k(k+1)}$$

*Split the summand, as the hint suggests.*

$$\frac{1}{k}-\frac{1}{k+1}=\frac{(k+1)-k}{k(k+1)}=\frac{1}{k(k+1)},$$

so the summand is $f(k)-f(k+1)$ with $f(k)=\dfrac{1}{k}$.

*Write out enough terms.*

$$\underbrace{\left(1-\tfrac{1}{2}\right)}_{k=1}+\underbrace{\left(\tfrac{1}{2}-\tfrac{1}{3}\right)}_{k=2}+\cdots+\underbrace{\left(\tfrac{1}{40}-\tfrac{1}{41}\right)}_{k=40}$$

*Apply the formula with $m=1$ and $n=40$.*

$$\sum_{k=1}^{40}\frac{1}{k(k+1)}=f(1)-f(41)=1-\frac{1}{41}$$

$$\boxed{\sum_{k=1}^{40} \frac{1}{k(k+1)} = \frac{40}{41} \approx 0.9756}$$

Letting the upper limit grow, the sum tends to 1. This is the standard example of a convergent infinite series, and you will meet it again in Chapter 10.

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**In short.** Whenever a summand is a difference of two similar expressions, test whether the second is the first with $k$ replaced by $k+1$. If it is, do not expand: write down $f(m) - f(n+1)$ and stop.