

Polynomial Long Division

The division step in Example 3, Section 8.5

MATH 111 — Integral Calculus • Dr. Sumayah Mansour Bin Hazzaa

Why we divide first. Partial fractions only work on a *proper* fraction, one whose numerator has *strictly* smaller degree than its denominator. In Example 3 we must integrate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx,$$

and the numerator has degree 3 while the denominator has degree 2. The fraction is *improper*, so no decomposition exists. We divide first, and only the leftover fraction is decomposed.

The layout

Write the dividend inside, the divisor outside, and the quotient on top. Leave a column for every power of x , highest on the left.

$$\begin{array}{r} \text{the quotient goes here} \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ \text{divisor} \qquad \qquad \text{dividend} \end{array}$$

Note: every power from x^3 down to the constant is present here. If one is missing, write it with coefficient 0 before you start.

Step 1. Divide the leading terms

Look only at the first term of each: $\frac{2x^3}{x^2} = 2x$. Write $2x$ above, in the x column.

Now multiply the *whole* divisor by $2x$:

$$2x(x^2 - 2x - 3) = 2x^3 - 4x^2 - 6x,$$

write it underneath, and subtract.

$$\begin{array}{r} \qquad \qquad \qquad 2x \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ - \quad 2x^3 - 4x^2 - 6x \\ \hline \qquad \qquad 0 \qquad 0 + 5x - 3 \end{array}$$

The subtraction, term by term:

$$(2x^3 - 2x^3) + (-4x^2 + 4x^2) + (-x + 6x) - 3 = 5x - 3.$$

The sign trap. You subtract the *entire* line, so every sign in it changes. Here $-x - (-6x) = -x + 6x = +5x$. Students who write $-7x$ have subtracted only the first term. Writing the minus sign in front of a bracket, $-(2x^3 - 4x^2 - 6x)$, prevents this.

Step 2. Stop when the degree drops

The new remainder is $5x - 3$, of degree 1. The divisor $x^2 - 2x - 3$ has degree 2.

Since $1 < 2$, we cannot divide again — and we stop. *That is the whole stopping rule:* keep going while the degree of what is left is at least the degree of the divisor.

Result of the division	
Quotient	$2x$
Remainder	$5x - 3$

Step 3. Write the answer in the standard form

Every division can be written as

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

So

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

The left-over fraction $\frac{5x - 3}{x^2 - 2x - 3}$ is now *proper*, degree 1 over degree 2, so partial fractions apply to it. That is where the lecture continues.

Always check

Multiply back: quotient $\times$ divisor + remainder should return the dividend.

$$\begin{aligned} 2x(x^2 - 2x - 3) + (5x - 3) &= 2x^3 - 4x^2 - 6x + 5x - 3 \\ &= 2x^3 - 4x^2 - x - 3 \quad \checkmark \end{aligned}$$

This takes ten seconds and catches every sign error.

A second example, with two terms in the quotient

Divide $x^3 + 2x^2 - 5$ by $x^2 + x - 2$.

The x term is missing from the dividend, so write $0x$ in its column.

$$\begin{array}{r}
 \overline{x^3 + 2x^2 + 0x - 5} \\
 - x^3 + x^2 - 2x \\
 - 5 \\
 - x^2 + x - 2 \\
 x - 3
 \end{array}$$

First term: $x^3/x^2 = x$. After subtracting, what is left is $x^2 + 2x - 5$, still of degree 2, so we continue.
 Second term: $x^2/x^2 = 1$. After subtracting, $x - 3$ has degree $1 < 2$, so we stop.

$$\frac{x^3 + 2x^2 - 5}{x^2 + x - 2} = x + 1 + \frac{x - 3}{x^2 + x - 2}$$

Three mistakes to avoid

1. **Forgetting a missing power.** In $x^3 + 2x^2 - 5$ there is no x term. Write $0x$, or your columns will shift and every later line will be wrong.
2. **Subtracting only the first term.** The minus sign applies to the whole line. Bracket it before you subtract.
3. **Stopping too early or too late.** Compare degrees, not the number of terms. Continue while $\deg(\text{what is left}) \geq \deg(\text{divisor})$.

Practice

Divide, and write each answer in the form quotient + remainder/divisor.

	Divide	Answer
(a)	$\frac{x^2 + 3x + 1}{x + 2}$	$x + 1 - \frac{1}{x + 2}$
(b)	$\frac{3x^3 + 2x^2 - 1}{x^2 + 1}$	$3x + 2 - \frac{3x + 3}{x^2 + 1}$
(c)	$\frac{2x^4 + x}{x^2 - 1}$	$2x^2 + 2 + \frac{x + 2}{x^2 - 1}$

In (b) and (c) a power is missing from the dividend — write it with coefficient 0 first. In every case, check by multiplying back.