الحل النموذجي

College of Science.
Department of Mathematics

جامعة الملك سعود
King Saud University
كلية العلوم
قسم الرياضيات

Final Exam
Academic Year 1444-1445 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	Integral Calculus		اسم المقرر
Course Code	Math 111 ريض 111		رمز المقرر
Exam Date	2023-12-13	1445-05-29	تاريخ الامتحان
Exam Time	08: 00 AM		وقت الامتحان
Exam Duration	3 hours	ثلاث ساعات	مدة الامتحان
Classroom No.	G043		رقم قاعة الاختبار
Instructor Name	حنان العوهلي		اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 8 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 8 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
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هذا الجزء خاص بأستاذ المادة
This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	CLO 2.1	QII	14	
2	CLO 2.2	QI	7	
3	CLO 2.3	QIII	4	
4	CLO 2.4	QIV+V	15	
5				40
6				
7				
8				

EXAM COVER PAGE

Question Number	I	II	III	IV	V	Total
Mark						

Question I:

A. Find the value of z that satisfies the conclusion of the Integral Mean Value Theorem for $f(x) = (x-1)^2 on\ [1,4]$. [3 points]

$$\int_a^b f(x)\,dx = f(z)(b-a)$$

$$\int_1^4 (x-1)^2\,dx = \frac{(x-1)^3}{3}\Big]_1^4 = \frac{3^3}{3} - 0 = 9$$

$$3(z-1)^2 = 9 \quad (z-1)^2 = 3 \qquad z-1 = \pm\sqrt{3}$$

$$\boxed{z = 1+\sqrt{3} \in (1,4)}$$

$$z = 1-\sqrt{3} \notin (1,4)$$

B. If $F(x) = \int_{ln|x|}^{e^x} \sqrt{t^2+5}\,dt$ then compute $F'(x)$. [2 points]

$$F'(x) = \sqrt{e^{2x}+5}\,(e^x) - \sqrt{(ln|x|)^2+5}\,\left(\frac{1}{x}\right)$$

C. Compute $f'(x)$ if

$$f(x) = sinh^{-1}(3^x) + ln\,(|tanh(4x)|)$$ [2 points]

$$f'(x) = \frac{3^x \ln 3}{\sqrt{3^{2x}+1}} + \frac{1}{\tanh(4x)} \cdot \operatorname{sech}^2(4x)\ (4)$$

Question II:

Evaluate the following integrals:

1. $\int \frac{2}{\sqrt{-x^2-6x}}\,dx$ [3 points]

$-(x^2+6x) = -(x^2+6x+3^2-3^2)$

$= -((x+3)^2-9) = 9-(x+3)^2$

$2\int \frac{1}{\sqrt{9-(x+3)^2}}\,dx = 2\int \frac{1}{\sqrt{9-u^2}}\,du$ $u = x+3$, $du = dx$

$= 2\sin^{-1}\frac{u}{3} + C = 2\sin^{-1}\frac{(x+3)}{3} + C$

2. $\int x^2 \cosh x\,dx$ [3 points]

$u = x^2$ $dv = \cosh x$

$du = 2x\,dx$ $v = \sinh x\,dx$

$I = x^2\sinh x - 2\int x\sinh x\,dx$, $u_1 = x$ $dv_1 = \sinh x\,dx$

$du_1 = dx$ $v_1 = \cosh x$

$= x^2\sinh x - 2(x\cosh x - \int \cosh x\,dx)$

$= x^2\sinh x - 2x\cosh x + 2\sinh x + C$

3. $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$ [3 points]

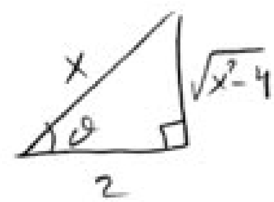
$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\,d\theta$ $\boxed{\sec\theta = \frac{x}{2}}$

$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta$

$I = \int \frac{1}{4\sec^{\cancel{2}}\theta\cdot\cancel{2}\cancel{\tan\theta}}\cdot\cancel{2}\cancel{\sec\theta}\cdot\cancel{\tan\theta}\;d\theta$

$= \frac{1}{4}\int\frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int\cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$

$= \frac{1}{4}\,\frac{\sqrt{x^2-4}}{x} + C$



4. $\int \frac{4x^2-x+12}{x^3+4x}\,dx$ [3 points]

$\frac{4x^2-x+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$= \frac{A(x^2+4)+(Bx+C)x}{x(x^2+4)} \Rightarrow 4x^2-x+12 = (A+B)x^2 + Cx + 4A$

$\therefore\ 4A = 12 \quad \boxed{A=3}\,,\ \boxed{C=-1} \qquad A+B=4 \Rightarrow \boxed{B=1}$

$I = \int\frac{3}{x}\,dx + \int\frac{x-1}{x^2+4}\,dx = 3\ln|x| + \frac{1}{2}\int\frac{2x}{x^2+4}\,dx - \int\frac{1}{x^2+4}\,dx$

$= 3\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

5. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$ [2 points]

$u = \sqrt[6]{x} \qquad u^6 = x \qquad 6u^5\,du = dx$

$u^3 = \sqrt{x} \quad \& \quad u^2 = \sqrt[3]{x}$

$I = \int \frac{1}{u^3+u^2}\,6u^5\,du = 6\int \frac{u^{\cancel{5}\,3}}{\cancel{u^2}(u+1)}\,du$

$= 6\int \frac{u^3}{u+1}\,du$

$= 6\int u^2-u+1\,du + 6\int \frac{-1}{u+1}\,du$

$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u\right] - 6\ln|u+1| + C$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x}+1| + C$

$$\begin{array}{r|l} & u^2 - u + 1 \\ u+1 & u^3 \\ & -(u^3+u^2) \\ & -u^2 \\ & \pm u^2 \pm u \\ & u \\ & -u \mp 1 \\ & -1 \end{array}$$

Question III:

A. Compute the following limit $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{e^x+5x}{e^{2x}+2x+1}.$$ [2 points]

$= \lim_{x\to\infty} \frac{e^x+5}{2e^{2x}+2} \quad \frac{\infty}{\infty} \quad = \lim_{x\to\infty} \frac{e^x}{4e^{2x}}$

$= \frac{1}{4}\lim_{x\to\infty} e^{-x} = \frac{1}{4}(0) = 0$

B. Determine whether the improper integral $\int_1^{\infty} \frac{1}{(2x-1)^3}\, dx$ converges or diverges. If it converges find its value. [2 points]

$$\lim_{t\to\infty} \frac{1}{2}\int_1^t \frac{2\cdot 1}{(2x-1)^3}\, dx = \lim_{t\to\infty} \frac{1}{2} \left.\frac{(2x-1)^{-2}}{-2}\right]_1^t$$

$$= -\frac{1}{4} \lim_{t\to\infty} \left[\frac{1}{(2t-1)^2} - 1 \right] = -\frac{1}{4}(0-1) = \frac{1}{4}$$

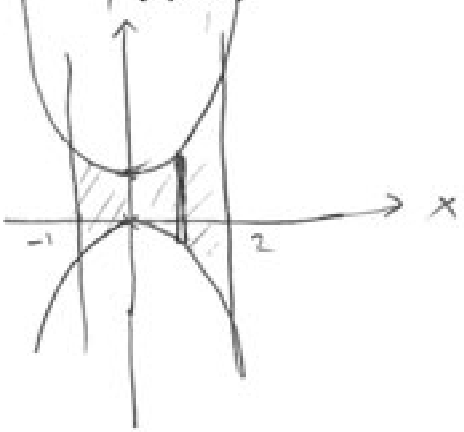
converges

Question IV:

A. Sketch the region R bounded by the graphs of the functions

$y = -x^2,\ y = x^2 + 1, x = -1$ *and* $x = 2.$

Then **find its area.** [3 points]



$$y = x^2 + 1$$
$$y - 1 = x^2$$

$$A = \int_{-1}^{2} x^2 + 1 - (-x^2)\, dx$$

$$= \int_{-1}^{2} 2x^2 + 1\, dx = \left.\frac{2}{3}x^3 + x\right]_{-1}^{2}$$

$$= \frac{2}{3}(8) + 2 - \left(\frac{-2}{3} - 1\right) = 9$$

B. **<u>Sketch</u>** the region R bounded by the graphs of

$$y = x^2 \text{ and } y = \sqrt{x}.$$

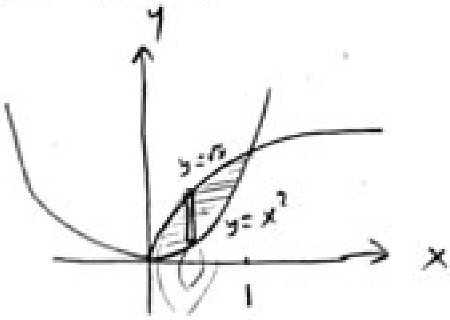
Then **find the volume** of the solid generated by revolving R about the $x - axis$. [3 points]

نقاط التقاطع

$\boxed{x^2 = \sqrt{x}}$ $\quad x^4 = x$

$x^4 - x = 0 \quad x(x^3 - 1) = 0$

$x = 0 \quad x = 1$



$$V = \pi \int_0^1 (\sqrt{x})^2 - (x^2)^2 \, dx$$

$$= \pi \int_0^1 (x - x^4) \, dx$$

$$= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \pi \left[\frac{1}{2} - \frac{1}{5} \right] = \frac{3\pi}{10}$$

C. Find the arc length of the graph of $y = 2 + \cosh(x)$ from $x = 0$ to $x = \ln 2$. [3 points]

$$L = \int_a^b \sqrt{1+\left(\frac{df}{dx}\right)^2}\, dx \qquad y' = \frac{df}{dx} = \sinh x$$

$$= \int_0^{\ln 2} \sqrt{1+\sinh^2 x}\, dx = \int_0^{\ln 2} \sqrt{\cosh^2 x}\, dx$$

$$= \int_0^{\ln 2} \cosh x\, dx = \sinh x \Big]_0^{\ln 2} = \sinh(\ln 2) - \sinh 0$$

$$= \frac{e^{\ln 2} - e^{-\ln 2}}{2} - \frac{e^0 - e^0}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}$$

Question V:

A. Find an equation in x *and* y that has the same graph as the polar equation [2 points]

$$r = 8\cos\theta + 6\sin\theta.$$

$$r^2 = 8r\cos\theta + 6r\sin\theta$$

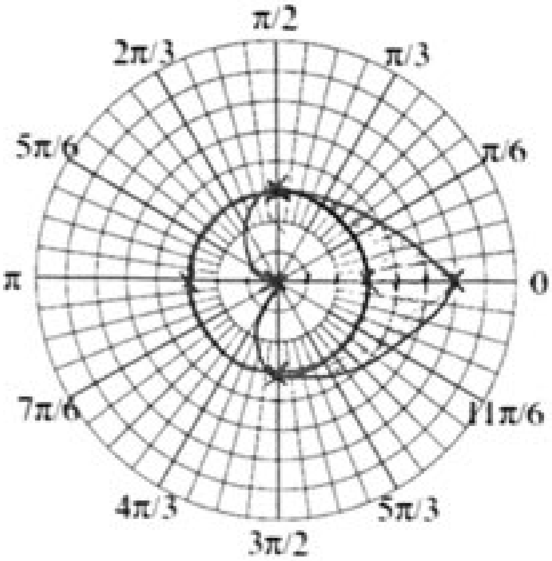
$$x^2 + y^2 = 8x + 6y$$

B. **Sketch** the region inside graph of the polar equation $r = 3 + 3\,cos\theta$ and outside the graph of the curve $r = 3$. Then **compute its area**. [4 points]

نقاط التقاطع

$3 = 3 + 3\cos\theta$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$



r	0	$\pi/2$	π	$3\pi/2$	2π
θ	6	3	0	3	6

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2} (3+3\cos\theta)^2 - 3^2 \quad d\theta$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\int_0^{\pi/2} \cancel{9} + 18\cos\theta + 9\cos^2\theta - \cancel{9} \quad d\theta$$

$$= \int_0^{\pi/2} 18\cos\theta + 9\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 18\sin\theta + \frac{9}{2}\theta + \frac{9}{2}\left(\frac{1}{2}\right)\sin 2\theta\Big]_0^{\pi/2}$$

$$= 18\sin\frac{\pi}{2} + \frac{9}{2}\left(\frac{\pi}{2}\right) + \frac{9}{4}\overset{0}{\cancel{\sin\pi}} - 0$$

$$= 18 + \frac{9\pi}{4}$$

Good Luck ☺