Question Number	Q1	Q2	Q3	Total
Mark				

Question 1: (3+2+2)

3 1. Find the value of c that satisfies the Mean Value Theorem for definite integrals for the function $f(x) = \frac{1}{\sqrt{x+1}}$ on the interval [3,8].

0.25 $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

0.25 $\frac{1}{\sqrt{c+1}} = \frac{1}{8-3}\int_3^8 \frac{1}{\sqrt{x+1}}\,dx$ 0.25 0.25 0.25

$= \frac{1}{5}\cdot 2\,(x+1)^{\frac{1}{2}}\Big|_3^8$ 0.25 0.25

$= \frac{2}{5}\left(9^{\frac{1}{2}} - 4^{\frac{1}{2}}\right)$ 0.25

$= \frac{2}{5}(3-2)$

$= \frac{2}{5}$ 0.25

0.25

$\Rightarrow \sqrt{c+1} = \frac{5}{2} \Rightarrow c+1 = \frac{25}{4} \Rightarrow c = \frac{25}{4} - \frac{4}{4} = \frac{21}{4} \approx 5.25 \in (3,8)$ 0.25 0.25

2 2. If $F(x) = \int_{sinhx}^{2^{x+1}} (1+t^2)dt$, find $F'(x)$.

$F'(x) = \left[1 + \left(2^{x+1}\right)^2\right] 2^{x+1}\ln 2 - \left[1 + \sinh^2 x\right]\cosh x$

0.25 0.25 0.25 0.25 0.25 0.25 0.25 0.25

2 3. If $y(x) = (cosx)^x + e^{-5x}$, find $\frac{dy}{dx}$.

$y = e^{\ln(\cos x)^x} + e^{-5x} = e^{x\ln\cos x} + e^{-5x}$

$\Rightarrow y'(x) = e^{x\ln\cos x}\cdot\left[1.\ln(\cos x) + x\,\frac{1}{\cos x}(-\sin x)\right] + e^{-5x}.(-5)$

0.25 0.25 0.25 0.25 0.25 0.25 0.25 0.25

$= (\cos x)^x\left[\ln(\cos x) - x\tan x\right] - 5e^{-5x}$

Question 2: (2+3+2+3+3+2)

Evaluate the following integrals:

2 1. $\int \frac{1}{\sqrt{x^2-4x}}\,dx$

0.25 0.25 0.25

$$x^2-4x = x^2-4x+4-4 = (x-2)^2-4$$

0.25

$$\int \frac{1}{\sqrt{x^2-4x}}\,dx = \int \frac{1}{\sqrt{(x-2)^2-4}}\,dx = \cosh^{-1}\left(\frac{x-2}{2}\right)+C$$

0.25 0.25 0.25

3 2. $\int x^2 \cos(3x)dx$

0.25 0.25

$u = x^2, \quad dv = \cos 3x\,dx$

$du = 2x\,dx \quad v = \frac{\sin 3x}{3}$

0.25 0.25

0.25 0.25 0.25 0.25

$$= x^2\,\frac{\sin 3x}{3} - \int 2x\,\frac{\sin 3x}{3}\,dx$$

$$= \frac{x^2}{3}\sin 3x - \frac{2}{3}\int x \sin 3x\,dx$$

$u = x, \quad dv = \sin 3x\,dx$

$du = dx \quad v = -\frac{\cos 3x}{3}$

0.25 0.25 0.25

$$= \frac{x^2}{3}\sin 3x - \frac{2}{3}\left[-\frac{x}{3}\cos 3x + \int \frac{\cos 3x}{3}\,dx\right]$$

0.25

$$= \frac{x^2}{3}\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{9}\,\frac{\sin 3x}{3} + C$$

3. $\int \frac{(\sqrt{x}-1)\sec(\sqrt{x}-\ln\sqrt{x})}{x}\,dx$

$$u = \sqrt{x} - \ln\sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow du = \frac{\sqrt{x}-1}{2\sqrt{x}\sqrt{x}}\,dx \Rightarrow du = \frac{\sqrt{x}-1}{2x}\,dx$$

$$\Rightarrow 2\,du = \frac{\sqrt{x}-1}{x}\,dx$$

$$\int \frac{(\sqrt{x}-1)\sec(\sqrt{x}-\ln\sqrt{x})}{x}\,dx = 2\int \sec u\,du$$

$$= 2\ln|\sec u + \tan u| + C = 2\ln\left|\sec(\sqrt{x}-\ln\sqrt{x}) + \tan(\sqrt{x}-\ln\sqrt{x})\right| + C$$

4. $\int \frac{5x^2+6x+4}{(x+1)(x^2+x+1)}\,dx$

$$\frac{5x^2+6x+4}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$= \frac{A(x^2+x+1) + (Bx+C)(x+1)}{(x+1)(x^2+x+1)}$$

$$\Rightarrow 5x^2+6x+4 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (A+B+C)x + A + C$$

$$\Rightarrow A+B=5, \quad A+B+C=6, \quad A+C=4$$

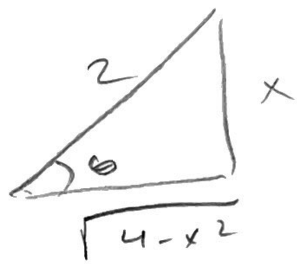
$$B = 5-A \qquad C = 4-A$$

$$\cancel{A} + 5 - \cancel{A} + 4 - A = 6 \Rightarrow A = 9-6 = 3 \Rightarrow B = 2,\ C = 1$$

5. $\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\, dx$

$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$\frac{x}{2} = \sin\theta$



$\int \frac{4\sin^2\theta}{(4-4\sin^2\theta)^{3/2}}\, 2\cos\theta\, d\theta$

$= \int \frac{\cancel{8}\sin^2\theta\cos\theta}{\cancel{8}(1-\sin^2\theta)^{3/2}}\, d\theta$

$= \int \frac{\sin^2\theta\cancel{\cos\theta}}{(\cos^{\cancel{2}}\theta)^{\cancel{3/2}}}\, d\theta$

$= \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = \int \tan^2\theta\, d\theta = \int \sec^2\theta - 1\, d\theta$

$= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\frac{x}{2} + C$

6. $\int tan^3x\, sec^2x\, dx$

$= \int \tan^2\cancel{x}\sec x\, \tan x \sec x\, dx$

$= \int (\sec^2 x - 1)(\sec x)\tan x \sec x\, dx$

$u = \sec x;\ du = \sec x \tan x\, dx$

$= \int (u^2-1)\, u\, du$

$= \int u^3 - u\, du$

$= \frac{u^4}{4} - \frac{u^2}{2} + C = \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + C$

Question 3: (2+2+3+3+2+2+4)

1. Calculate $\lim_{x\to\infty} x^{\frac{1}{x}}$.

$$y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\ln \infty}{\infty} = \frac{\infty}{\infty}.$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0.$$

$$\Rightarrow \lim_{x\to\infty} y = e^{0} = 1$$

2. Determine whether the improper integral $\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$ converges or diverges.

$$\int_0^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{r\to\infty} \int_0^r \frac{x}{(1+x^2)^2}\,dx$$

$$= \lim_{r\to\infty} \frac{1}{2} \int_0^r 2x\,(1+x^2)^{-2}\,dx.$$

$$= \lim_{r\to\infty} \frac{1}{2} \left.\frac{(1+x^2)^{-1}}{-1}\right|_0^r.$$

$$= -\frac{1}{2} \lim_{r\to\infty} \left.\frac{1}{1+x^2}\right|_0^r$$

$$= -\frac{1}{2} \lim_{r\to\infty} \left(\frac{1}{1+r^2} - \frac{1}{1}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{\infty} - 1\right)$$

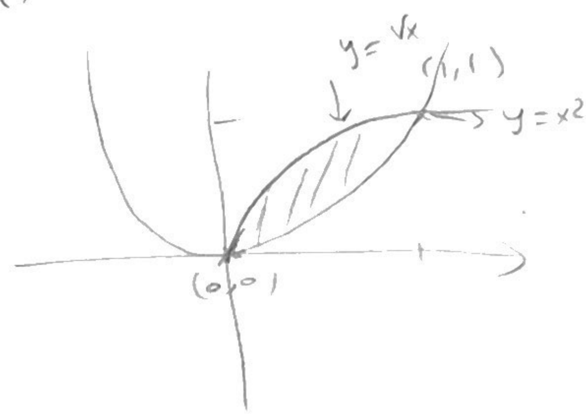
$$= -\frac{1}{2}(0-1) = \frac{1}{2}$$

Converges

3. **Sketch** the region bounded by the graphs of the curves $y = x^2, y = \sqrt{x}$.**Then find its area.**

$x^2 = \sqrt{x} \Rightarrow x^4 = x \Rightarrow x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$

$(0,0)$, $(1,1)$



$$\text{Area} = \int_0^1 \sqrt{x} - x^2 \, dx$$

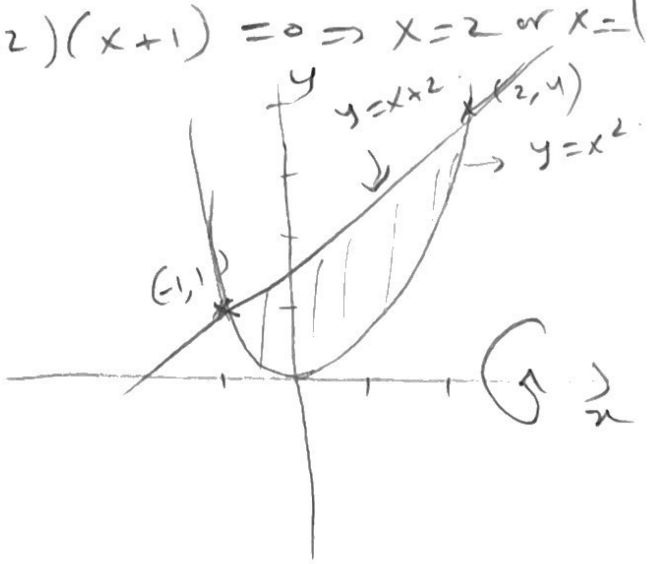
$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

4. **Sketch** the region bounded by the graphs of the curves $y = x^2,\ y = x + 2$. **Then find the volume** of the solid generated by revolving this region about the ***x − axis***.

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

$(2,4)$, $(-1,1)$



$$\text{volume} = \int_{-1}^{2} \pi \left[(x+2)^2 - (x^2)^2 \right] dx$$

$$= \pi \int_{-1}^{2} x^2 + 4x + 4 - x^4 \, dx$$

$$= \pi \left(\frac{x^3}{3} + 4\frac{x^2}{2} + 4x - \frac{x^5}{5} \right) \Big|_{-1}^{2}$$

$$= \pi \left[\left(\frac{8}{3} + 8 + 8 - \frac{32}{5} \right) - \left(-\frac{1}{3} + 2 - 4 + \frac{1}{5} \right) \right]$$

$$= \pi (\quad)$$

5. Find the arc length of $y = \frac{2}{3}x\sqrt{x}$ from $x = 0$ to $x = 1$.

$y = \frac{2}{3}x^{\frac{3}{2}} \Rightarrow y' = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} x^{\frac{1}{2}} \Rightarrow (y')^2 = (x^{\frac{1}{2}})^2 = x$

$$\therefore L = \int_0^1 \sqrt{1 + (y')^2}\, dx$$

$$= \int_0^1 \sqrt{1 + x}\, dx$$

$$= \frac{2}{3}(1 + x)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{2}{3}\left(2^{\frac{3}{2}} - 1\right)$$

6. Convert the polar equation $r = cos\theta + sec\theta$ into a Cartesian equation.

$$r^2 = r\cos\theta + r\sec\theta$$

$$r^2 = r\cos\theta + \frac{r}{\sin\theta}$$

$$r^2 = r\cos\theta + \frac{r^2}{r\sin\theta}$$

$$x^2 + y^2 = x + \frac{x^2 + y^2}{y}$$

7. **Sketch** the common region between the curves $r = 2$ and $r = 2 + 2\cos\theta$. **Then find its area**.

$$\text{Area} = 2\left[\int_0^{\frac{\pi}{2}} \frac{1}{2}[2+2\cos\theta]^2\,d\theta + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\,2^2\,d\theta\right]$$

$$= \int_0^{\frac{\pi}{2}} 4+8\cos\theta+4\cos^2\theta\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 4\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4+8\cos\theta+\cancel{4}^{2}\,\frac{1+\cos2\theta}{\cancel{2}}\,d\theta + \int_{\frac{\pi}{2}}^{\pi} 4\,d\theta$$

$$= 4\theta + 8\sin\theta + 2\theta + \cancel{2}\,\frac{\sin2\theta}{\cancel{2}}\Big|_0^{\frac{\pi}{2}} + 4\theta\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \cancel{6}\,\frac{\pi}{\cancel{2}} + 8\sin\frac{\pi}{2} + 4\left(\pi - \frac{\pi}{2}\right)$$

$$= 3\pi + 8 + 2\pi = 5\pi + 8$$