Q1: Let us consider raw elementary operations on the corresponding augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & \lambda^2-5 & \lambda \end{array}\right) \begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array} \equiv \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{array}\right) \begin{array}{l} \\ \frac{1}{2}R_2 \\ R_3-\frac{\lambda^2-4}{2}R_2 \end{array}$$

$$\equiv \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & \lambda-\frac{\lambda^2}{2} \end{array}\right)$$

Notice that $\lambda - \frac{\lambda^2}{2} = -\frac{1}{2}\lambda(\lambda-2)$

① If $\lambda = 0$ or 2; the system has infinitely many solutions

② If $\lambda \neq 0$ and $\lambda \neq 2$; the system has no solution

③ Because there is no leading 1 on the second column, the system can never have a unique solution

Q2: To find the inverse of the coefficient matrix of the system:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) R_2-R_1 \equiv \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1-R_3 \\ R_2 \rightleftarrows R_3 \\ \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & -1 & 1 & 0 \end{array}\right) R_3+2R_2 \equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 & 1 & 2 \end{array}\right) \begin{array}{l} R_2-\frac{1}{3}R_3 \\ \frac{1}{3}R_3 \end{array}$$

$$\equiv \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{array}\right)$$

Hence, the coefficient matrix is invertible and its inverse is:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1/3 & -1/3 & 1/3 \\ -1/3 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

The solution of the system is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Q3: We have:

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \begin{matrix} R_2-R_1 \\ = \\ R_3-R_1 \\ R_4-R_1 \end{matrix} \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix} \begin{matrix} R_3-R_2 \\ R_4-R_2 \end{matrix}$$

$$= \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{vmatrix} R_4-R_3 = \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{vmatrix}$$

$$= a(b-a)(c-b)(d-c)$$

Q4) The comatrix of A is given by

$$Com\ A = \begin{pmatrix} -14 & 0 & 7 \\ 3 & -2 & -2 \\ -9 & -1 & 6 \end{pmatrix}$$

Therefore, $Adj\ A = Com\ A^T = \begin{pmatrix} -14 & 3 & -9 \\ 0 & -2 & -1 \\ 7 & -2 & 6 \end{pmatrix}$

Moreover, $\det A = -24 + 0 + 3 + 18 - 4 - 0$
$= -7 \neq 0.$

Hence, the matrix A is invertible.

Q5) The system can be written

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We have

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = 0 - 3 + 2 - 0 - 3 - 2 = -6 \neq 0$. This is a Cramer' system.

$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = 0 - 3 + 0 - 0 - 3 - 0 = -6$. Thus $x = \frac{-6}{-6} =$

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & -1 \end{vmatrix} = 0 + 3 + 2 - 0 - 3 + 2 = 4$. Thus $y = \frac{4}{-6} = -$

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0 + 0 + 2 - 0 - 0 + 2 = 4$. Thus $z = \frac{4}{-6} =$

The unique solution of the system is $(1 ; -\frac{2}{3} ; -\frac{2}{3})$.