Question 1

(a) Use Riemann sums to find $\int_0^2 (x^2+4)dx$. [3 points]

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_k = \frac{2k}{n}$$

$$R_p = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left[\left(\frac{2k}{n}\right)^2+4\right]\frac{2}{n} = \sum_{k=1}^{n}\left(\frac{4k^2}{n^2}\cdot\frac{2}{n}+\frac{8}{n}\right)$$

$$= \frac{8}{n^3}\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n}\frac{8}{n}$$

$$= \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + 8$$

$$= \frac{4(n+1)(2n+1)}{3n^2} + 8$$

$$\int_0^2 (x^2+4)\,dx = \lim_{n\to\infty} R_p$$

$$= \lim_{n\to\infty}\left[\frac{4(n+1)(2n+1)}{3n^2}+8\right]$$

$$= \frac{8}{3}+8$$

$$= \frac{32}{3}$$

(b) Evaluate the integrals:

i. $\displaystyle\int \frac{\left(1+x^{\frac{2}{3}}\right)^4}{x^{\frac{1}{3}}}dx.$ [2 points]

$u = 1 + x^{2/3}$

$du = \frac{2}{3} x^{-1/3} dx$

$\frac{3}{2} du = \frac{1}{x^{1/3}} dx$

$= \frac{3}{2}\int u^4 \, du$

$= \frac{3}{2} \cdot \frac{u^5}{5} + C$

$= \frac{3}{10} (1 + x^{2/3})^5 + C$

ii. $\displaystyle\int \frac{\ln x + 1}{\sqrt{9 - x^2(\ln x)^2}}dx.$ [2 points]

$= \int \frac{\ln x + 1}{\sqrt{9 - (x \ln x)^2}} dx$

$u = x \ln x$

$du = (\ln x + x \cdot \frac{1}{x}) dx$

$= (\ln x + 1) \, dx$

$= \int \frac{1}{\sqrt{9 - u^2}} du$

$= \sin^{-1} \frac{u}{3} + C$

$= \sin^{-1} \frac{x \ln x}{3} + C.$

iii. $\displaystyle\int \frac{dx}{x\sqrt{1-x^8}}.$ [3 points]

$$= \int \frac{x^3}{x^4\sqrt{1-x^8}}\,dx$$

$u = x^4$

$du = 4x^3\,dx$

$\frac{1}{4}\,du = x^3\,dx$

$$= \frac{1}{4}\int \frac{1}{u\sqrt{1-u^2}}\,du$$

$$= \frac{1}{4}\,\operatorname{sech}^{-1}|u| + C$$

$$= \frac{-1}{4}\,\operatorname{sech}^{-1} x^4 + C.$$

iv. $\displaystyle\int x\sec^2 x\,dx.$ [3 points]

$u = x$ $\quad dv = \sec^2 x\,dx$

$du = dx$ $\quad v = \tan x$

$$= x\tan x - \int \tan x\,dx$$

$$= x\tan x - (-\ln|\cos x|) + C$$

$$= x\tan x + \ln|\cos x| + C$$

(or $x\tan x - \ln|\sec x| + C$)

v. $\int \sin^2 x \cos^5 x dx.$ [3 points]

$$= \int \sin^2 x \cos^4 x \cos x \, dx$$

$$= \int \sin^2 x (1 - \sin^2 x)^2 \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^2 (1 - u^2)^2 \, du$$

$$= \int u^2 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{3} u^3 - \frac{2}{5} u^5 + \frac{1}{7} u^7 + C = \frac{1}{3} \sin^3 x - \frac{2}{5} \sin^5 x + \frac{1}{7} \sin^7 x + C.$$

vi. $\int \frac{x+3}{(x^2+9)^{\frac{3}{2}}} dx.$ [3 points]

$x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta \, d\theta$, $(x^2+9)^{3/2} = (9\tan^2\theta + 9)^{3/2} = 3^3 \sec^3\theta$

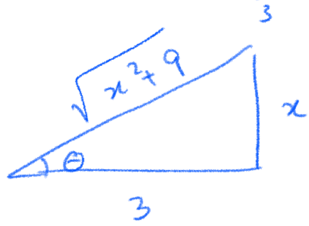
$$\therefore \int \frac{x+3}{(x^2+9)^{3/2}} dx = \int \frac{3\tan\theta + 3}{3^3 \sec^3\theta} \cdot 3\sec^2\theta \quad \theta$$

$$= \frac{1}{3} \int \frac{\tan\theta + 1}{\sec\theta} \, d\theta$$

$$= \frac{1}{3} \int (\sin\theta + \cos\theta) \, d\theta$$

$\frac{x}{3} = \tan\theta$

$$= \frac{1}{3} (-\cos\theta + \sin\theta) + C$$



$$= \frac{1}{3} \left(-\frac{3}{\sqrt{x^2+9}} + \frac{x}{\sqrt{x^2+9}} \right) + C$$

vii. $\displaystyle\int \frac{x^3+3}{(x+1)(x^2+1)}dx.$ [3 points]

$(x+1)(x^2+1) = x^3+x^2+x+1$

$$\begin{array}{r} 1 \\ x^3+x^2+x+1 \overline{\big) \; x^3+3} \\ \ominus x^3 \ominus x^2 \ominus x \ominus 1 \\ \hline -x^2-x+2 \end{array}$$

$$= \int 1 + \frac{-x^2-x+2}{(x+1)(x^2+1)}\, dx$$

$$\frac{-x^2-x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\therefore \; -x^2-x+2 = A(x^2+1) + (Bx+C)(x+1)$$

$x=-1 \Rightarrow \quad -1+1+2 = 2A \Rightarrow \boxed{A=1}$

Comparing Coefficients x^2: $-1 = A+B \Rightarrow \boxed{B=-2}$

x: $-1 = B+C \Rightarrow \boxed{C=1}$

$$\therefore \int \frac{x^3+3}{(x+1)(x^2+1)}\, dx = \int \left(1 + \frac{1}{x+1} + \frac{-2x-1}{x^2+1}\right) dx$$

$$= x + \ln|x+1| - \ln(x^2+1) - \tan^{-1}x + C.$$

Question 2

(a) Compute $\lim_{x \to +\infty} (1 + e^{-2x})^{e^x}$. [3 points]

$y = (1+e^{-2x})^{e^x}$

$\ln y = e^x \ln(1+e^{-2x})$

$= \dfrac{\ln(1+e^{-2x})}{e^{-x}}$

$\lim_{x \to +\infty} \ln y = \lim_{x \to +\infty} \dfrac{\ln(1+e^{-2x})}{e^{-x}} \quad \left(\frac{0}{0}\right)$

$= \lim \dfrac{\frac{-2e^{-2x}}{1+e^{-2x}}}{-e^{-x}} = \lim_{x\to\infty} \dfrac{-2e^{-2x}}{-e^{-x}(1+e^{-2x})} = \lim_{x\to\infty} \dfrac{2e^{-x}}{1+e^{-2x}} = 0$

$\therefore \lim y = e^0 = 1.$

(b) Find $\displaystyle\int_3^{+\infty} \frac{dx}{x(\ln x)^3}$. [3 points]

$= \lim_{t\to\infty} \displaystyle\int_3^t \frac{dx}{x(\ln x)^3}$ $\qquad u = \ln x, \ du = \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \displaystyle\int_{\ln 3}^{\ln t} u^{-3}\,du$

$= \lim_{t\to\infty} \left.\dfrac{u^{-2}}{-2}\right]_{\ln 3}^{\ln t}$

$= -\frac{1}{2} \lim_{t\to\infty} \left[\dfrac{1}{(\ln t)^2} - \dfrac{1}{(\ln 3)^2}\right]$

$= \dfrac{1}{2(\ln 3)^2}.$

Question 3

(a) Find the surface area obtained by revolving the curve $x = t^3$, $y = 3t + 1$, $0 \leq t \leq 1$ about the $y-$axis. [3 points]

$$S = \int_0^1 2\pi t^3 \sqrt{(3t^2)^2 + 3^2}\; dt$$

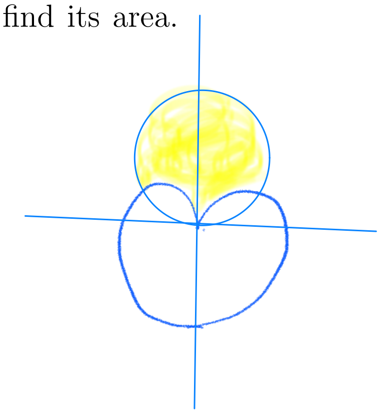
$$= \int_0^1 6\pi t^3 \sqrt{t^4+1}\; dt$$

$u = t^4 + 1$
$du = 4t^3\, dt$

$$= \int_1^2 \frac{6\pi}{4} \sqrt{u}\; du$$

$$= \frac{3}{2}\pi \cdot \frac{2}{3} u^{\frac{3}{2}} \Big]_1^2 = \pi\left[2^{\frac{3}{2}} - 1\right].$$

(b) Sketch the region inside $r = 3\sin\theta$ and outside $r = 3 - 3\sin\theta$ and find its area. [3 points]



$$3\sin\theta = 3 - 3\sin\theta$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \quad \therefore \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\therefore \quad A = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\left[3^2\sin^2\theta - (3 - 3\sin\theta)^2\right] d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\sin^2\theta - 9 + 18\sin\theta - 9\sin^2\theta \; d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 18\sin\theta - 9 \; d\theta = \left(-18\cos\theta - 9\theta\right)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

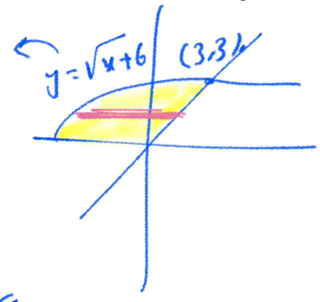
$$= \left(0 - 9\left(\frac{\pi}{2}\right)\right) - \left(-18\frac{\sqrt{3}}{2} - 9\frac{\pi}{6}\right)$$

$$= -\frac{9\pi}{2} + 9\sqrt{3} + \frac{3\pi}{2} = -\frac{6\pi}{2} + 9\sqrt{3} = -3\pi + 9\sqrt{3}.$$

Question 4

(a) Sketch the region bounded by the curves $y = \sqrt{x+6}$, the $x-$axis, $y = x$, and find its area. [3 points]

$x = y^2 - 6$



points of intersection

$\sqrt{x+6} = x \Rightarrow x + 6 = x^2$

$\Rightarrow x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$\therefore$ $(3,3)$ is the pt of intersection

$$A = \int_0^3 y - (y^2 - 6)\, dy = \frac{y^2}{2} - \frac{y^3}{3} + 6y \Big]_0^3$$

$$= \frac{9}{2} - 9 + 18$$

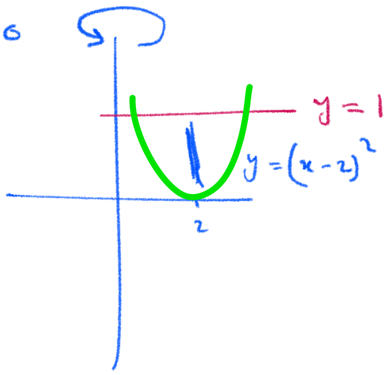
$$= \frac{9}{2} + 9.$$

(b) Find the volume of the solid obtained by revolving the region bounded by $y = (x-2)^2$, $y = 1$ about the $y-$axis. [3 points]

$(x-2)^2 = 1 \Rightarrow x^2 - 4x + 4 = 1$

$x^2 - 4x + 3 = 0$ $\quad \therefore (x-3)(x-1) = 0$

$x = 3,\ x = 1$



$$V = \int_1^3 2\pi x \,(1 - (x-2)^2)\, dx$$

$$= \int_1^3 2\pi x \,(1 - (x^2 - 4x + 4))\, dx$$

$$= \int_1^3 2\pi x \,(-x^2 + 4x - 3)\, dx$$

$$= 2\pi \int_1^3 (-x^3 + 4x^2 - 3x)\, dx$$

$$= 2\pi \left(-\frac{x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right)\Big]_1^3$$

$$= 2\pi \left[\left(-\frac{3^4}{4} + \frac{4}{3}3^3 - \frac{3}{2}3^2\right) - \left(-\frac{1}{4} + \frac{4}{3} - \frac{3}{2}\right)\right]$$