King Saud University

College of Science.
Department of Mathematics

كلية العلوم
قسم الرياضيات

Final Exam
Academic Year 1446 Hijri- SecondSemester

Exam Information معلومات الامتحان			
Course name	Integral Calculus حساب التكامل		اسم المقرر
Course Code	106 ريض		رمز المقرر
Exam Date	2025-05-21	1446-11-24	تاريخ الامتحان
Exam Time	01: 00 PM		وقت الامتحان
Exam Duration	3 hours	ثلاث ساعات	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
nstructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
tudent's Name		اسم الطالب
D number		الرقم الجامعي
ection No.		رقم الشعبة
erial Number		الرقم التسلسلي

eneral Instructions: **تعليمات عامة:**

- Your Exam consists of 8 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
- Calculators are not allowed.

- عدد صفحات الامتحان 8 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
- الآلة الحاسبة ممنوعة.

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
.1	I		
.2	IIA		
.3	IV		
4	II(B+C)		
5	III		

Question Number	I	IIA	II(B+C)	III	IV	Total
Mark						

Question I: : [3+2+3+3+3+3+2=19 points]

Evaluate the following integrals:

1. Use the substitution $u = 1 + x^2$ to compute $\int x^5\sqrt{1+x^2}\,dx$. [3 points]

$u = 1+x^2 \rightarrow x^2 = u-1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$I = \int x^4 \sqrt{1+x^2}\; x\,dx = \frac{1}{2}\int (u-1)^2 \sqrt{u}\,du$

$= \frac{1}{2}\int (u^2-2u+1)\sqrt{u}\,du = \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$

$= \frac{1}{2}\left[\frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + c$

$= \frac{u^{7/2}}{7} - \frac{2u^{5/2}}{5} + \frac{u^{3/2}}{3} + c$

$= \frac{(1+x^2)^{7/2}}{7} - \frac{2(1+x^2)^{5/2}}{5} + \frac{(1+x^2)^{3/2}}{3} + c.$

2. $\int \frac{1}{x\sqrt{4-(\ln x)^2}}\,dx =?$ [2 points]

$u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int \frac{1}{\sqrt{4-u^2}}\,du = \sin^{-1}\frac{u}{2} + c$

$= \sin^{-1}\frac{\ln x}{2} + c.$

3. $\int x \cos^2 x \, dx$ [3 points]

$u = x$ $\quad dv = \cos^2 x \, dx = \frac{1}{2}(1+\cos 2x)\, dx$

$du = dx$ $\quad v = \frac{1}{2} x + \frac{1}{4} \sin 2x$

$\int x \cos^2 x \, dx = \frac{1}{2} x^2 + \frac{1}{4} x \sin 2x - \int (\frac{1}{2} x + \frac{1}{4} \sin 2x)\, dx$

$= \frac{1}{2} x^2 + \frac{1}{4} x \sin 2x - \frac{1}{4} x^2 + \frac{1}{8} \cos 2x + c.$

$= \frac{1}{4} x^2 + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x + c.$

$\int \cos^5 x \sin^6 x \; dx = \int \cos^4 x \sin^6 x \cos x \, dx$ $\quad u = \sin x \to du = \cos x \, dx$ [3 points]

$= \int (1-\sin^2 x)^2 \sin^6 x \cos x \, dx$

$= \int (1-u^2)^2 u^6 \, du$

$= \int (1-2u^2+u^4) u^6 \, du$

$= \int (u^6 - 2u^8 + u^{10})\, du$

$= \frac{u^7}{7} - \frac{2}{9} u^9 + \frac{u^{11}}{11} + c$

$= \frac{\sin^7 x}{7} - \frac{2}{9} \sin^9 x + \frac{\sin^{11} x}{11} + c.$

5. $\int \frac{x^2}{\sqrt{25-x^2}}\,dx = I$. $x = 5\sin\theta$ [3 points]

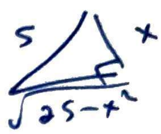
$dx = 5\cos\theta\, d\theta$

$\sqrt{25-x^2} = \sqrt{25-25\sin^2\theta} = 5\cos\theta$.

$\therefore I = \int \frac{25\sin^2\theta}{5\cos\theta} \cdot 5\cos\theta\, d\theta = 25\int \sin^2\theta\, d\theta$

$= \frac{25}{2}\int (1-\cos 2\theta)\, d\theta = \frac{25}{2}\theta - \frac{25}{4}\sin 2\theta + C$, as $\frac{x}{5} = \sin\theta \rightarrow \theta = \sin^{-1}\frac{x}{5}$

$= \frac{25}{2}\theta - \frac{25}{4} 2\sin\theta\cos\theta + C$



$\cos\theta = \frac{\sqrt{25-x^2}}{5}$

$= \frac{25}{2}\sin^{-1}\frac{x}{5} - \frac{25}{2}\frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5} + C$

$= \frac{1}{2}\sin^{-1}\frac{x}{5} - \frac{1}{2}x\sqrt{25-x^2} + C.$

6. $\int \frac{x+4}{x(x+1)^2}\,dx = I$ [3 points]

$\frac{x+4}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$x+4 = A(x+1)^2 + Bx(x+1) + Cx$

if $x=0$: $\boxed{4 = A}$

if $x=-1$: $3 = -C \therefore C = -3$ $\therefore B = -4$

coefficient of x^2: $0 = A + B \rightarrow B = -A$

$\therefore I = \int \left(\frac{4}{x} + \frac{-4}{x+1} + \frac{-3}{(x+1)^2}\right) dx$

$= 4\ln|x| - 4\ln|x+1| + \frac{3}{x+1} + C$

$= \ln\left(\frac{x}{x+1}\right)^4 + \frac{3}{x+1} + C.$

7. $\int \frac{1}{\sqrt{x^2+4x+3}}\,dx = I.$ $x^2+4x+3 = x^2+4x+(2)^2+3-(2)^2$ **[2 points]**

$= (x+2)^2 + 1$

$= \int \frac{1}{\sqrt{(x+2)^2+1}}\,dx.$, $u = x+2$

$du = dx$

$= \int \frac{1}{\sqrt{u^2+1}}\,du = \cosh^{-1} u + c$

$= \cosh^{-1}(x+2) + c.$

estion II: [3+3+3=9 points]

Find the number(s) of z that satisfies the conclusion of the Integral Mean Value Theorem for he definite integral $f(x) = \frac{1}{(x+1)^2}$ on the interval [0,3]. [3 points]

$f(z) = \frac{1}{3-0}\int_0^3 \frac{1}{(x+1)^2}\,dx = \frac{1}{3}\left[-\frac{1}{x+1}\right]_0^3$

$= -\frac{1}{3}\left[\frac{1}{4} - 1\right] = -\frac{1}{3}\left[-\frac{3}{4}\right] = \frac{1}{4}$

$\therefore \frac{1}{(z+1)^2} = \frac{1}{4}$

$\rightarrow (z+1)^2 = 4$

$z^2 + 2z + 1 - 4 = 0$

$z^2 + 2z - 3 = 0$

$(z+3)(z-1) = 0$

$z = -3 \notin (0,3)$

8. Find

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^{x^2}$$. of the form ∞^0 [3 points]

Put $y = \left(\frac{1}{x}\right)^{x^2}$ $\ln|y| = x^2 \ln\frac{1}{x}$ $\lim_{x\to 0^+} x^2 \ln\frac{1}{x}$ of the form $0(-\infty)$

$\therefore \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x^2 \ln\frac{1}{x} = \lim_{x\to 0^+}\frac{\ln\frac{1}{x}}{1/x^2}$ of the form $\frac{\infty}{\infty}$ Using L'Hopital's rule:

$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{-\ln x}{1/x^2} = \lim_{x\to 0^+}\frac{-\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0^+}\frac{-1}{x}\cdot\frac{x^3}{-2} = \lim_{x\to 0} -\frac{x^2}{2}$

$= 0$

$\therefore \lim_{x\to 0^+}\ln y = 0 \rightarrow \lim_{x\to 0^+} y = e^0 = 1$

Determine whether the improper integral

$$\int_0^\infty xe^{-2x^2}\,dx$$

converges or diverges. If it converges find its value. [3 points]

$\int_0^\infty x e^{-2x^2} dx = \lim_{t\to\infty}\int_0^t x e^{-2x^2} dx$

$= -\frac{1}{4}\lim_{t\to\infty}\int_0^t -4x e^{-2x^2} dx$

$= -\frac{1}{4}\lim_{t\to\infty} e^{-2x^2}\Big]_0^t$

$= -\frac{1}{4}\lim_{t\to\infty}\left[e^{-2t^2} - e^0\right]$

$= -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{e^{2t^2}} - 1\right] = -\frac{1}{4}(0-1) = \frac{1}{4}$

$\therefore$ The improper integral converges to $\frac{1}{4}$.

uestion III: [3+3=6 points]

. Sketch the region R bounded by the graphs of the functions

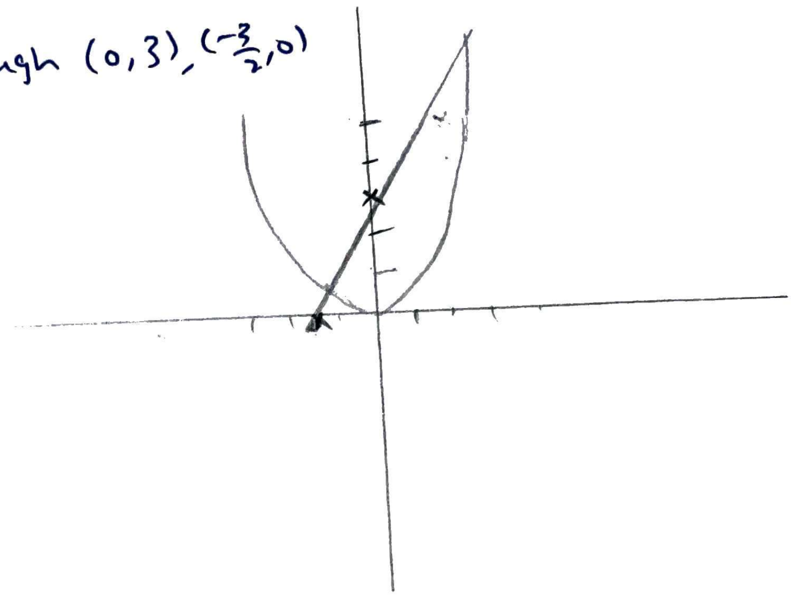
$$y = x^2, \qquad y = 2x + 3\,.$$

[3 points]

hen **find its area**.

$=x^2$: parabola with vertex $(0,0)$ open upwards.

$x+3$: straight line passing through $(0,3)$, $(-\frac{3}{2},0)$



ts of intersection:

$$x^2 = 2x+3$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$= 3\,, \quad x = -1$$

$$3,9)\,, \ (-1,1)$$

$$ea = \int_{-1}^{3} \left[(2x+3) - x^2\right] dx$$

$$= \left.x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$

$$= (9+9-9) - \left(1-3-\frac{1}{3}\right) = 9 - \left(-2-\frac{1}{3}\right)$$

$$= 11 - \frac{1}{3} = \frac{33-1}{3} = \frac{32}{3}\,.$$

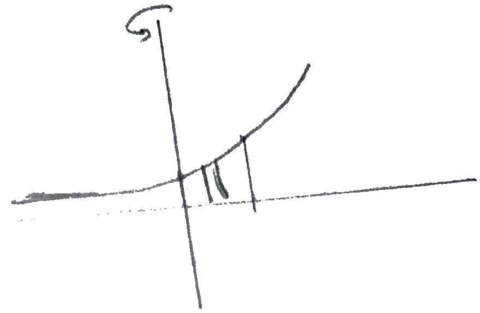
B. **Find the volume** of the solid obtained by revolving the region bounded by

$y = e^{2x}$, $\quad y = 0$, $\quad x = 0$ *and* $x = 1$,

[3 points]

about the $y -$ *axis*.

$V = 2\pi \int_0^1 x\, e^{2x}\, dx$



$u = x \qquad dv = e^{2x}\, dx$

$du = dx \qquad v = \frac{1}{2} e^{2x}$

$\therefore V = 2\pi \left[\frac{x e^{2x}}{2} \Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}\, dx \right]$

$= 2\pi \left[\left(\frac{e^2}{2} - 0 \right) - \frac{1}{4} e^{2x} \right]_0^1$

$= 2\pi \left[\frac{e^2}{2} - \frac{1}{4} e^2 + \frac{1}{4} \right] = 2\pi \left[\frac{e^2}{4} + \frac{1}{4} \right] = \frac{\pi}{2}(e^2+1).$

Question IV: [3+3=6 points]

A. Find the arc length of the parametric curve

$x = \sin t - t\cos t, \qquad y = \cos t + t\sin t\ , 0 \le t \le 2.$

[3 points]

$\frac{dx}{dt} = \cos t - \cos t + t\sin t = t\sin t.$

$\frac{dy}{dt} = -\sin t + \sin t + t\cos t = t\cos t.$

$\therefore L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^2 \sqrt{t^2 \sin^2 t + t^2 \cos^2 t}\, dt = \int_0^2 \sqrt{t^2 \cdot 1}\, dt \quad , \ t \in [$

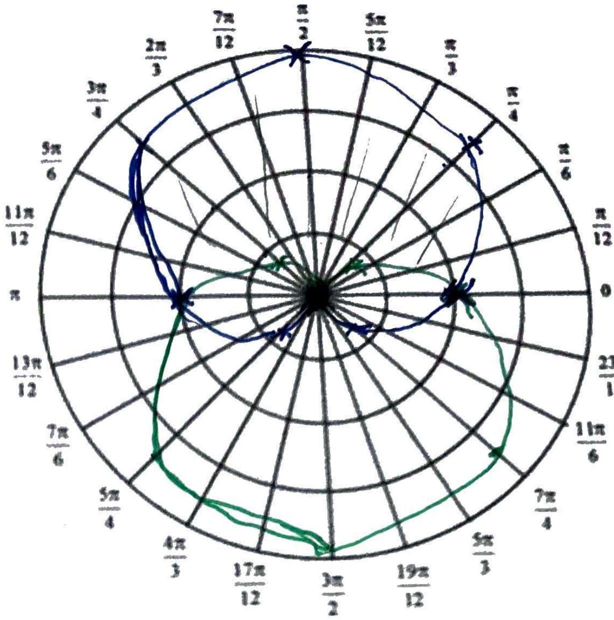
$= \int_0^2 t\, dt = \frac{t^2}{2} \Big|_0^2 = \frac{4}{2} = 2.$

graph of the curve $r = 2 - 2\sin\theta$. Then **find its area**. [3 points]

Interval	$\sin\theta$	$r_1 = 2+2\sin\theta$	$r_2 = 2 - 2\sin\theta$
$[0, \frac{\pi}{2}]$	$0 \to 1$	$2 \to 4$	$2 \to 0$
$[\frac{\pi}{2}, \pi]$	$1 \to 0$	$4 \to 2$	$0 \to 2$
$[\pi, \frac{3\pi}{2}]$	$0 \to -1$	$2 \to 0$	$2 \to 4$
$[\frac{3\pi}{2}, 2\pi]$	$-1 \to 0$	$0 \to 2$	$4 \to 2$

$r_1(\frac{\pi}{4}) = 2 + \sqrt{2} \simeq 3.4$

$r_2(\frac{\pi}{4}) = 2 - \sqrt{2} \simeq 0.6$



Points of intersection:

$$2 + 2\sin\theta = 2 - 2\sin\theta$$

$$4\sin\theta = 0$$

$$\sin\theta = 0 \ , \quad \theta = 0, \pi.$$

$$= \frac{1}{2}\int_0^{\pi} (2+2\sin\theta)^2 - (2-2\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2} \cdot 2 \cdot \int_0^{\frac{\pi}{2}} [[4 + 8\sin\theta + 4\sin^2\theta] - [4 - 8\sin\theta + 4\sin^2\theta]] \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} [16\sin\theta] \, d\theta = -16\cos\theta \Big]_0^{\frac{\pi}{2}}$$

$$= -16[-1-(+1)] = -16[-2] = 16.$$

Good Luck