King Saud University: Mathematics Department Math-254
Second Semester 1446 H Second Midterm Exam. Solution
Maximum Marks = 25 Time: 90 mins.

Questions: (6 + 7 + 6 + 6) Marks

Question 1:

Use Gauss elimination by partial pivoting to find the inverse A^{-1} involving α ($\neq 2$) of the following matrix,

$$A = \begin{pmatrix} 1 & \alpha & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then use A^{-1} with $\alpha = 3$ to solve the given linear system $A\mathbf{x} = [2, 2, 3]^T$.

Solution. Suppose that the inverse $A^{-1} = B$ of the given matrix exists and let

$$AB = \begin{pmatrix} 1 & \alpha & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Now to find the elements of the matrix B, we apply the Gaussian elimination using partial pivoting on the augmented matrix

$$[A|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 1 & \alpha & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right).$$

For the first elimination step, since 1 is the largest absolute coefficient of first variable x_1, therefore, no need to row interchange, using the possible multiples $m_{21} = 1$ and $m_{31} = 1$, gives

$$\equiv \left(\begin{array}{ccc|ccc} 1 & \alpha & 1 & 1 & 0 & 0 \\ 0 & (2-\alpha) & 1 & -1 & 1 & 0 \\ 0 & (2-\alpha) & 2 & - & 0 & 1 \end{array}\right).$$

Now use the multiple $m_{32} = 1$, gives

$$\equiv \left(\begin{array}{ccc|ccc} 1 & \alpha & 1 & 1 & 0 & 0 \\ 0 & (2-\alpha) & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) = [U|C].$$

Obviously, the original set of equations has been transformed to an equivalent upper-triangular form. We solve the first upper triangular linear system

$$[U|\mathbf{b}_1] = \begin{pmatrix} 1 & \alpha & 1 \\ 0 & (2-\alpha) & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = c_1,$$

by using backward substitution, we get

$$\begin{array}{rcrcrcr} b_{11} & + & \alpha b_{21} & + & b_{31} & = & 1 \\ & & (2-\alpha)b_{21} & + & b_{31} & = & -1 \\ & & & & b_{31} & = & 0 \end{array}$$

which gives, $\mathbf{b_1} = [b_{11}, b_{21}, b_{31}]^T = [2/(2-\alpha), 1/(\alpha-2), 0]^T$. Similarly, the solution of the second upper triangular linear system

$$[U|\mathbf{b}_2] = \begin{pmatrix} 1 & \alpha & 1 \\ 0 & (2-\alpha) & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = c_2,$$

can be obtained by using backward substitution, as

$$\begin{array}{rcrcrcr} b_{12} & + & \alpha b_{22} & + & b_{32} & = & 0 \\ & & (2-\alpha)b_{22} & + & b_{32} & = & 1 \\ & & & & b_{32} & = & -1 \end{array}$$

which gives, $\mathbf{b_2} = [b_{12}, b_{22}, b_{32}]^T = [(2-3\alpha)/((2-\alpha), 2/(2-\alpha), -1]^T$. Finally, the solution of the third upper triangular linear system

$$[U|\mathbf{b}_3] = \begin{pmatrix} 1 & \alpha & 1 \\ 0 & (2-\alpha) & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_3,$$

can be obtained by using backward substitution, as

$$\begin{array}{rcrcrcr} b_{13} & + & \alpha b_{23} & + & b_{33} & = & 0 \\ & & (2-\alpha)b_{23} & + & b_{33} & = & 0 \\ & & & & b_{33} & = & 1 \end{array}$$

which gives, $\mathbf{b_3} = [b_{13}, b_{23}, b_{33}]^T = [(2(1-\alpha)/(\alpha-2), 1/(\alpha-2), 1]^T$. Hence the elements of the inverse matrix B are

$$B = A^{-1} = \begin{pmatrix} \frac{2}{(2-\alpha)} & \frac{(2-3\alpha)}{(2-\alpha)} & \frac{2(1-\alpha)}{(\alpha-2)} \\ \\ \frac{1}{(\alpha-2)} & \frac{2}{(2-\alpha)} & \frac{1}{(\alpha-2)} \\ \\ 0 & -1 & 1 \end{pmatrix},$$

which is the required inverse of the given matrix A involving α.
Thus using $\alpha = 3$ and A^{-1}, we can solve the linear system as

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -2 & 7 & -4 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},$$

gives the required solution of the given system. •

Question 2:

Use LU decomposition by Dollittle's method to find the value(s) of $\alpha(\neq 2)$ for which the following matrix

$$A = \begin{pmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

is singular. Compute the unique solution of the linear system $A\mathbf{x} = [1, 0, -1/2]^T$ by using the largest negative integer value of α.

Solution. Since we know that

$$A = \begin{pmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = \frac{\alpha}{2} = l_{21}$, $m_{31} = -\frac{1}{2} = l_{31}$, and $m_{32} = \dfrac{1}{(2-\alpha)} = l_{32} \quad (\alpha \neq 2)$, gives

$$\begin{pmatrix} 2 & \alpha & -1 \\ 0 & \frac{(4-\alpha^2)}{2} & \frac{(2+\alpha)}{2} \\ 0 & \frac{(2+\alpha)}{2} & \frac{1}{2} \end{pmatrix} \equiv \begin{pmatrix} 2 & \alpha & -1 \\ 0 & \frac{(4-\alpha^2)}{2} & \frac{(2+\alpha)}{2} \\ 0 & \frac{(2+\alpha)}{2} & -\frac{\alpha}{(2-\alpha)} \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\alpha}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{(2-\alpha)} & 1 \end{pmatrix} \begin{pmatrix} 2 & \alpha & -1 \\ 0 & \frac{(4-\alpha^2)}{2} & \frac{(2+\alpha)}{2} \\ 0 & 0 & -\frac{\alpha}{(2-\alpha)} \end{pmatrix} = LU,$$

which is the required decomposition of A. The matrix will be singular if

$$det(A) = det(U) = -\frac{2\alpha(4-\alpha^2)}{2(2-\alpha)} = -\alpha(2+\alpha) = 0, \quad \text{gives}, \quad \alpha = 0, \ \alpha = -2.$$

To find the unique solution of the given system we take $\alpha = -1$ and it gives

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [1, 1/2, -1/6]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ -\frac{1}{6} \end{pmatrix}.$$

Performing backward substitution yields, $[x_1, x_2, x_3]^T = [1/2, 1/2, -1/2]^T$, the solution. •

Question 3:

Consider the following linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 3 \\ x_1 & + & 3x_2 & + & x_3 & = & 5 \\ & - & 2x_2 & + & 3x_3 & = & 1 \end{array}$$

Find the Gauss-Seidel iteration matrix T_G and show that $\|T_G\|_\infty > 1$. Use Gauss-Seidel method to compute the approximate solution of the system within accuracy 5×10^{-2}, using the initial solution $\mathbf{x}^{(0)} = [0.9, 0.9, 0.9]^T$. If $\mathbf{x} = [1, 1, 1]^T$ is the exact solution of the linear system, then compute the absolute error.

Solution. Since the Gauss-Seidel iteration matrix is defined as

$$T_G = -(L+D)^{-1}U = -\begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 2/3 & -1/3 \\ 0 & 4/9 & -2/9 \end{pmatrix}.$$

Then the l_∞ norm of the matrix T_G is

$$\|T_G\|_\infty = \max\left\{2, 1, \frac{2}{3}\right\} = 2 > 1.$$

The Gauss-Seidel iterative formula for the given system is defined as

$$\begin{array}{rcrcrcl} x_1^{(k+1)} & = & (3 & - & 2x_2^{(k)} & &) \\ x_2^{(k+1)} & = & 1/3(5 & - & x_1^{(k+1)} & - & x_3^{(k)}) \\ x_3^{(k+1)} & = & 1/3(1 & & & + & 2x_2^{(k+1)}) \end{array}$$

Starting with initial approximation $\mathbf{x}^{(0)} = [0.9, 0.9, 0.9]^T$, we have

$$\mathbf{x}^{(1)} = [1.2000, 0.9667, 0.9778]^T, \quad \|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|_\infty = 0.3000 \nleq 5 \times 10^{-2},$$

$$\mathbf{x}^{(2)} = [1.0667, 0.9852, 0.9901]^T, \quad \|\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\|_\infty = 0.1333 \nleq 5 \times 10^{-2},$$

$$\mathbf{x}^{(3)} = [1.0296, 0.9934, 0.9956]^T, \quad \|\mathbf{x}^{(3)} - \mathbf{x}^{(2)}\|_\infty = 0.0371 \leq 5 \times 10^{-2}.$$

The absolute error is

$$\|\mathbf{x} - \mathbf{x}^{(3)}\|_\infty = \|[1,1,1]^T - [1.0296, 0.9934, 0.9956]^T\|_\infty = \|[-0.0296, 0.0066, 0.0044]^T\|_\infty = 0.0296.$$

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Question 4:

Consider a nonsingular linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 4/3 & 1 & -8/3 \\ -1/3 & 0 & 2/3 \\ -2/3 & -1 & 7/3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

If $\mathbf{b}$ of the given linear system is changed to $\mathbf{b}^* = [1, 1, 1.99]^T$, then how large a relative error can this change produce in the solution to the given linear system $A\mathbf{x} = \mathbf{b}$?

Solution. Given the matrix A and its inverse as

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 4/3 & 1 & -8/3 \\ -1/3 & 0 & 2/3 \\ -2/3 & -1 & 7/3 \end{pmatrix}.$$

Then

$$\|A\|_\infty = 5, \|A^{-1}\|_\infty = 5,$$

and we obtained the condition number of the given matrix as

$$K(A) = \|A\|_\infty \||A^{-1}\|_\infty = (5)(5) = 25.$$

The residual vector can be calculated as

$$\mathbf{r} = \mathbf{b} - A\mathbf{x}^* = \mathbf{b} - \mathbf{b}^* = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1.99 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.01 \end{pmatrix},$$

gives,

$$\|\mathbf{r}\|_\infty = 0.01.$$

Since the relative error formula is

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|_\infty}{\|\mathbf{x}\|_\infty} \leq K(A) \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty},$$

so using $K(A) = 25$, $\|\mathbf{r}\|_\infty = 0.01$ and the value $\|\mathbf{b}\|_\infty = 2$, we obtain

$$\frac{\|\mathbf{x} - \mathbf{x}^*\|_\infty}{\|\mathbf{x}\|_\infty} \leq (25) \frac{(0.01)}{2} = 0.1250,$$

the possible relative change in the solution to the given linear system. •