King Saud University: **Mathematics Department** **Math-254**
Third Semester **1444 H** **Solution of Midterm Exam.**
Maximum Marks = 30 **Time: 120 mins.**

Question 1: Show that the one of the possible rearrangement of the nonlinear equation $e^x - x^2 = 2 - 3x$, which has a root in the interval $[0, 0.6]$ is

$$x = g(x) = \frac{2 - e^x + x^2}{3}.$$

Show that $g(x)$ has a unique fixed-point in $[0, 0.6]$. Determine the number of iterations needed to get an approximation with accuracy 10^{-5} to the solution of

$$x_{n+1} = g(x_n) = \frac{2 - e^x_n + x_n^2}{3}; \qquad n \geq 0,$$

lying in the interval $[0, 0.6]$ by using the fixed-point iteration method and $x_0 = 0.5$. [6 Marks]

Solution. Since $f(x) = e^x - x^2 + 3x - 2$, we observe that $f(0)f(0.6) = (-1.0000)(1.2621) < 0$, then the solution we seek is in the interval $[0, 0.6]$. Given $g(x) = \dfrac{2 - e^x + x^2}{3}$ is continuous in $[0, 0.6]$ and

$$g(0) = \frac{2 - e^0 + 0}{3} = 0.3333 \quad \text{and} \quad g(0.6) = \frac{2 - e^{0.6} + (0.6)^2}{3} = 0.1793.$$

Also, g is decreasing function of x, and both lie in the interval $[0, 0.6]$. Thus $g(x) \in [0, 0.6]$, for all $x \in [0, 0.6]$. Also, we have $g'(x) = \dfrac{-e^x + 2x}{3}$ continuous in $(0, 0.6)$ and

$$g'(0) = \frac{-e^0 + 2(0)}{3} = -0.3333 \quad \text{and} \quad g'(0.6) = \frac{-e^{0.6} + 2(0.6)}{3} = -0.2074.$$

Also, g' is decreasing function of x, and both lie in the interval $(0, 0.6)$. Thus $|g'(x)| < 1$, for all x in the given interval $(0, 0.6)$, so from fixed-point theorem the g(x) has a unique fixed-point. Using the given initial approximation $x_0 = 0.5$, we have the first approximation as

$$x_1 = g(x_0) = \frac{2 - e^{x_0} + x_0^2}{3} = \frac{2 - e^{0.5} + (0.5)^2}{3} = 0.2004.$$

Since $a = 0$ and $b = 0.6$, then the value of k can be found as follows

$$k_1 = |g'(0)| = 0.3333 \quad \text{and} \quad k_2 = |g'(0.6| = 0.2074,$$

which give $k = \max\{k_1, k_2\} = 0.3333$. Thus using the error bound formula,

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0|, \qquad \text{for all} \quad n \geq 1,$$

and given

$$|\alpha - x_n| \leq 10^{-5}, \quad \text{so we have} \quad \frac{k^n}{1-k}|x_1 - x_0| \leq 10^{-5},$$

$$\frac{(0.3333)^n}{1 - 0.3333}|0.2004 - 0.5| \leq 10^{-5}, \quad n \ \ln(0.3333) \leq \ln(2.2252 \times 10^{-5}), \quad \text{gives}, \quad n \geq 9.7506,$$

So we need ten (10) approximations to get the desired accuracy for the given problem. •

Question 2: Show that the rate of convergence of Newton's method at the root $\alpha = 1$ of the equation $(x-1)^2 \sin x = 0$ is linear. Use quadratic convergence method to find x_2 using $x_0 = 1.5$. Compute the relative error. [6 Marks]

Solution. Since

$$f(x) = (x-1)^2 \sin x \quad \text{and} \quad f'(x) = 2(x-1)\sin x + (x-1)^2 \cos x,$$

and $f'(1) = 0$, gives that $\alpha = 1$ is the multiple root. Using Newton's iterative formula, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{((x_n-1)^2 \sin x_n)}{(2(x_n-1)\sin x_n + (x_n-1)^2 \cos x_n)} = x_n - \frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)},$$

for $n \geq 0$. The fixed point form of the developed Newton's formula is

$$x_{n+1} = g(x_n) = x_n - \frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)}.$$

Then

$$g(x) = x - \frac{((x-1)\sin x)}{(2\sin x + (x-1)\cos x)},$$

and

$$g'(x) = 1 - \frac{(2\sin x + (x-1)\cos x)(\sin x + (x-1)\cos x) - ((x-1)\sin x)(3\cos x - (x-1)\sin x)}{(2\sin x + (x-1)\cos x)^2}.$$

Thus

$$g'(1) = 1 - \frac{2(\sin 1)^2}{4(sin 1)^2} = \frac{1}{2} \neq 0,$$

and so the Newton's method converges linearly. The quadratic convergent method for multiple root is modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0,$$

where m is the order of multiplicity of the zero of the function. To find m, we check that

$$f''(x) = 2\sin x + 4(x-1)\cos x - (x-1)^2 \sin x, \quad \text{and} \quad f''(1) = 2\sin 1 \neq 0,$$

so $m = 2$. Thus

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{((x_n-1)\sin x_n)}{(2\sin x_n + (x_n-1)\cos x_n)}, \qquad n \geq 0.$$

Now using initial approximation $x_0 = 1.5$, we have the following two approximations

$$x_1 = x_0 - 2\frac{((x_0-1)\sin x_0)}{(2\sin x_0 + (x_0-1)\cos x_0)} = 1.0087, \quad x_2 = x_1 - 2\frac{((x_1-1)\sin x_1)}{(2\sin x_1 + (x_1-1)\cos x_1)} = 1.0000,$$

and the relative error is,

$$\frac{|\alpha - x_2|}{|\alpha|} = \frac{|1 - 1.0000|}{1} = 0.0000.$$

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Question 3: Find the values of a, b and c such that the iterative scheme

$$x_{n+1} = ax_n + \frac{bN}{x_n^2} + \frac{cN^2}{x_n^5}, \qquad n \geq 0,$$

converges at least cubically to $\alpha = N^{\frac{1}{3}}$. Use this scheme to find second approximation of $(8)^{\frac{1}{3}}$ when $x_0 = 1.8$. [6 Marks]

Solution. Given the iterative scheme converges at least cubically means $g' = g'' = 0$ at $\alpha = N^{\frac{1}{3}}$. Let

$$g(x) = ax + \frac{bN}{x^2} + \frac{cN^2}{x^5}, \quad g(N^{\frac{1}{3}}) = 1 = a + b + c,$$

$$g'(x) = a - \frac{2bN}{x^3} - \frac{5cN^2}{x^6}, \quad g'(N^{\frac{1}{3}}) = 0 = a - 2b - 5c,$$

$$g''(x) = 0 + \frac{6bN}{x^4} + \frac{30cN^2}{x^7}, \quad g''(N^{\frac{1}{3}}) = 0 = 3b + 15c,$$

Solving these three equations for unknowns a, b and c, we obtain $a = b = \frac{5}{9}$ and $c = -\frac{1}{9}$. Thus

$$x_{n+1} = \frac{5x_n}{9} + \frac{5N}{9x_n^2} - \frac{N^2}{9x_n^5}, \qquad n \geq 0.$$

Using $x_0 = 1.8,\ N = 8$, we obtain

$$x_1 = \frac{5x_0}{9} + \frac{5N}{9x_0^2} - \frac{N^2}{9x_0^5} = 1.9954,$$

and

$$x_2 = \frac{5x_1}{9} + \frac{5N}{9x_1^2} - \frac{N^2}{9x_1^5} = 2.0000,$$

the required second approximation.
The absolute error can be obtained as

$$|(8)^{1/3} - 2.0000| = |2.0000 - 2.0000| = 0.0000 \quad \text{up to 4 dp.}$$

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Question 4: Use the simple Gaussian elimination method to find the inverse A^{-1} of the following matrix

$$A = \begin{pmatrix} 1 & -1 & 2\alpha \\ 2 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix}.$$

Using A^{-1}, find the unique solution of the system $A\mathbf{x} = [-4, 8, 12]^T$ taking α the largest possible negative integer value. [6 Marks]

Solution. Suppose that the inverse $A^{-1} = B$ of the given matrix exists and let

$$AB = \begin{pmatrix} 1 & -1 & 2\alpha \\ 2 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Now to find the elements of the matrix B, we apply the simple Gaussian elimination on the augmented matrix

$$[A|\mathbf{I}] = \left(\begin{array}{ccc} 1 & -1 & 2\alpha \\ 2 & 2 & 4 \\ 0 & 4 & 8 \end{array} \vdots \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

Using $m_{21} = \dfrac{2}{1} = 2$ and $m_{31} = 0$, gives

$$[A|\mathbf{I}] \equiv \left(\begin{array}{ccc} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 4 & 8 \end{array} \vdots \begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

Using $m_{32} = \dfrac{4}{4} = 1$, gives

$$[A|\mathbf{I}] \equiv \left(\begin{array}{ccc} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{array} \vdots \begin{array}{ccc} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{array}\right).$$

Solve the first upper system

$$\begin{pmatrix} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},$$

by using backward substitution, we get

$$\begin{array}{rcrcrcr} b_{11} & - & b_{21} & + & 2\alpha b_{31} & = & 1 \\ & & 4b_{21} & + & (4-4\alpha)b_{31} & = & -2 \\ & & & + & (4+4\alpha)b_{31} & = & 2 \end{array}$$

which gives $b_{11} = 0$, $b_{21} = -\frac{1}{1+\alpha}$, $b_{31} = \frac{1}{2(1+\alpha)}$. Similarly, the solution of the second upper system

$$\begin{pmatrix} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcrcr} b_{12} & - & b_{22} & + & 2\alpha b_{32} & = & 0 \\ & & 4b_{22} & + & (4-4\alpha)b_{32} & = & 1 \\ & & & + & (4+4\alpha)b_{32} & = & -1 \end{array}$$

which gives $b_{12} = \frac{1+\alpha}{2(\alpha+1)}$, $b_{22} = \frac{1}{2(\alpha+1)}$, $b_{32} = -\frac{1}{4(\alpha+1)}$. Finally, the solution of the third upper system

$$\begin{pmatrix} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcrcr} b_{13} & - & b_{23} & + & 2\alpha b_{33} & = & 0 \\ & & 4b_{23} & + & (4-4\alpha)b_{33} & = & 0 \\ & & & + & (4+4\alpha)b_{33} & = & 1 \end{array}$$

and it gives $b_{13} = -\frac{1}{4}$, $b_{23} = \frac{\alpha-1}{4(\alpha+1)}$, $b_{33} = \frac{1}{4(\alpha+1)}$. Hence the elements of the inverse matrix B are Thus

$$B = A^{-1} = \begin{pmatrix} 0 & \frac{1+\alpha}{2(\alpha+1)} & -\frac{1}{4} \\ \\ -\frac{1}{1+\alpha} & \frac{1}{2(\alpha+1)} & \frac{\alpha-1}{4(\alpha+1)} \\ \\ \frac{1}{2(1+\alpha)} & -\frac{1}{4(\alpha+1)} & \frac{1}{4(\alpha+1)} \end{pmatrix},$$

which is the required inverse of the given matrix A for any α. The inverse does not exist at $\alpha = -1$, so take $\alpha = -2$, we get

$$B = A^{-1} = \begin{pmatrix} 0 & 0.5 & -0.25 \\ 1 & -0.5 & 0.75 \\ -0.5 & 0.25 & -0.25 \end{pmatrix},$$

which is the required inverse of the given matrix A. Now to find the solution of the system, we do as

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 0 & 0.5 & -0.25 \\ 1 & -0.5 & 0.75 \\ -0.5 & 0.25 & -0.25 \end{pmatrix} \begin{pmatrix} -4 \\ 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the required solution. •

Question 5: Use LU decomposition by Dollittle's method to find the value(s) of nonzero α for which the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix},$$

is inconsistent and consistent. Solve the consistent system. [6 Marks]

Solution. Since we know that

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = LU.$$

Using $m_{21} = \frac{2\alpha}{\alpha} = 2 = l_{21}$, $m_{31} = \frac{1}{\alpha} = l_{31}$, and $m_{32} = \dfrac{(3\alpha - 4)}{(-9\alpha)} = l_{32} (\alpha \neq 0)$, gives

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & \frac{(3\alpha-4)}{\alpha} & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \equiv \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form. Thus

$$A = \begin{pmatrix} \alpha & 4 & 1 \\ 2\alpha & -1 & 2 \\ 1 & 3 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix},$$

which is the required decomposition of A. The given linear system has no solution or infinitely many solution if

$$det(A) = det(U) = \frac{-9\alpha(\alpha^2 - 1)}{\alpha} = -9(\alpha^2 - 1) = (\alpha^2 - 1) = 0,$$

which gives, $\alpha = -1$ or $\alpha = 1$.
To find the solution of the given system when $\alpha = -1$ and it gives

$$\begin{pmatrix} -1 & 4 & 1 \\ -2 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{7}{9} & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{7}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, 4]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} -1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 4 \end{pmatrix}.$$

Last row gives, $0x_1 + 0x_2 + 0x_3 = 4$, which is not possible, and so no solution. To find the solution of the given system when $\alpha = 1$ and it gives

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & -1 & 2 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{9} & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now solving the lower-triangular system $L\mathbf{y} = \mathbf{b}$ for unknown vector $\mathbf{y}$, that is

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, 0]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 0 \end{pmatrix}.$$

Last row gives, $0x_1+0x_2+0x_3 = 0$, which means we have many solutions. Performing backward substitution and using $x_3 = t$, yields

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & x_3 & = & 6 \\ & & -9x_2 & & & = & -9 \end{array}$$

and it gives, $[x_1, x_2, x_3]^T = [2 - t, 1, t]^T$, for any nonzero t.

To find the unique solution for system for nonzero α ($\neq \pm 1$) we do as follows:

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{\alpha} & \frac{(3\alpha-4)}{(-9\alpha)} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} = \mathbf{b}.$$

Performing forward substitution yields, $[y_1, y_2, y_3]^T = [6, -9, \frac{2\alpha-2}{\alpha}]^T$.
Then solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ for unknown vector $\mathbf{x}$, that is

$$\begin{pmatrix} \alpha & 4 & 1 \\ 0 & -9 & 0 \\ 0 & 0 & \frac{(\alpha^2-1)}{\alpha} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ \frac{2\alpha-2}{\alpha} \end{pmatrix},$$

and performing backward substitution yields, $[x_1, x_2, x_3]^T = [\frac{2\alpha-2}{\alpha^2-1}, 1, \frac{2\alpha-2}{\alpha^2-1}]^T$. •