**King Saud University: Mathematics Department Math-254**
**First Semester 1444 H Midterm Exam.**
**Maximum Marks = 30 Time: 120 mins.**

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**Question 1:** Show that by eliminate $x^2$ between the equations $4x^2 + y^2 = 4$ and $x^2y^3 = 1$, we get a polynomial equation $y^5 - 4y^3 + 4 = 0$ in $y$. Use $y_0 = 1$ to find second approximation of the intersection point $(x, y)$ of the curves with $x > 0$ using Newton's method. [5 Marks]

**Solution.** Given

$$4x^2 + y^2 = 4 \qquad \text{and} \qquad x^2y^3 = 1.$$

Eliminate $x^2$, we have

$$x^2 = \frac{1}{y^3} \quad \text{or} \quad \frac{4}{y^3} + y^2 = 4, \quad \text{gives} \quad y^5 - 4y^3 + 4 = 0,$$

$$f(y) = y^5 - 4y^3 + 4 \qquad \text{and} \qquad f'(y) = 5y^4 - 12y^2.$$

Applying Newton's iterative formula to find the approximation of this polynomial equation, we have

$$y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} = y_n - \frac{y_n^5 - 4y_n^3 + 4}{5y_n^4 - 12y_n^2}.$$

Finding the second approximation using the initial approximation $y_0 = 1$, we get

$$y_1 = y_0 - \frac{y_0^5 - 4y_0^3 + 4}{5y_0^4 - 12y_0^2} = 1.1429,$$

$$y_2 = y_1 - \frac{y_1^5 - 4y_1^3 + 4}{5y_1^4 - 12y_1^2} = 1.1399,$$

and

$$x^2 = \frac{1}{y^3} \qquad \text{or} \qquad x = \frac{1}{y^{3/2}} = \frac{1}{(1.1399)^{3/2}} = 0.8217.$$

Thus the point of intersection is $(x, y) = (0.8217, 1.1399)$.

**Question 2:** The equation $(1 - x)^2 e^{1-x} = 0$ has multiple root $\alpha = 1$. Develop the Modified Newton's formula for computing the approximation of this root, then use it to find the second approximation $x_2$ using $x_0 = 0.75$. Show that the developed formula converges at least quadratically to the root. [5 Marks]

**Solution.** Since $\alpha = 1$ is a root of $f(x)$, so

$$\begin{array}{lcll} f(x) & = & (1-x)^2 e^{1-x}, & f(1) = 0, \\ f'(x) & = & -2(1-x)e^{1-x} - (1-x)^2 e^{1-x} = (x-3)(1-x)e^{1-x}, & f'(1) = 0, \\ f''(x) & = & (7 - 6x + x^2)e^{1-x}, & f''(1) = 2 \neq 0, \end{array}$$

the function has zero of multiplicity 2. Using modified Newton's iterative formula, we get

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{(1-x_n)^2 e^{1-x_n}}{(x_n - 3)(1-x_n)e^{1-x_n}} = x_n - 2\frac{(1-x_n)}{(x_n-3)}, \quad n \geq 0.$$

Now evaluating this at the give approximation $x_0 = 0.75$, gives

$$x_1 = x_0 - 2\frac{(1-x_0)}{(x_0-3)} = 0.9722 \quad \text{and} \quad x_2 = x_1 - 2\frac{(1-x_0)}{(x_1-3)} = 0.9996.$$

The fixed point form of the developed modified Newton's formula is

$$x_{n+1} = g(x_n) = x_n - 2\frac{(1-x_n)}{(x_n-3)},$$

where

$$g(x) = x - 2\frac{(1-x)}{(x-3)}.$$

By taking first derivative, we have

$$g'(x) = 1 - 2\left(\frac{2}{(x-3)^2}\right), \quad g'(1) = 1 - 1 = 0.$$

Thus the method converges at least quadratically to the given root.

**Question 3:** Consider the nonlinear system

$$\begin{array}{rcl} x^3 + y^3 & = & k_1 \\ -x + y^3 & = & k_2 \end{array}$$

Using Newton's method by taking the initial approximation $(x_0, y_0)^T = (0.5, 0.5)^T$ gives the first approximation $(x_1, y_1)^T = (1.0800, 0.4400)^T$. Find the values of $k_1$ and $k_2$. [5 Marks]

**Solution.** We are given the nonlinear system

$$\begin{array}{rcl} x^3 \quad + y & = & k_1 \\ -x \quad + y^3 & = & k_2 \end{array}$$

and it gives the functions and the first partial derivatives as follows:

$$\begin{array}{rcl} f_1(x,y) & = & x^3 + y^3 - k_1, \quad f_{1x} = 3x^2, \quad f_{1y} = 3y^2, \\ f_1(x,y) & = & -x + y^3 - k_2, \quad f_{2x} = -1, \quad f_{2y} = 3y^2. \end{array}$$

At the given initial approximation $x_0 = 0.5$ and $y_0 = 0.5$, we get

$$\begin{array}{rclrclrcl} f_1(0.5, 0.5) & = & 0.25 - k_1, & \dfrac{\partial f_1}{\partial x} = f_{1x} & = & 0.75, & \dfrac{\partial f_1}{\partial y} = f_{1y} & = & 0.75, \\ f_2(0.5, 0.5) & = & -0.3750 - k_2, & \dfrac{\partial f_1}{\partial x} = f_{2x} & = & -1, & \dfrac{\partial f_2}{\partial y} = f_{2y} & = & 0.75. \end{array}$$

The Jacobian matrix $J$ and its inverse $J^{-1}$ at the given initial approximation can be calculated as follows:

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 0.75 & 0.75 \\ -1 & 0.75 \end{pmatrix} \quad \text{and} \quad J^{-1} = \begin{pmatrix} 0.5714 & -0.5714 \\ 0.7619 & 0.5714 \end{pmatrix}.$$

Substituting all these values in Newton's formula, we get the first approximation as follows:

$$\begin{pmatrix} 1.0800 \\ 0.4400 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 0.5714 & -0.5714 \\ 0.7619 & 0.5714 \end{pmatrix} \begin{pmatrix} 0.25 - k_1 \\ -0.3750 - k_2 \end{pmatrix},$$

gives

$$\begin{array}{rcl} 0.5714k_1 - 0.5714k_2 & = & 0.9371 \\ 0.7619k_1 + 0.5714k_2 & = & -0.0838 \end{array}$$

then solving for $k_1$ and $k_2$, we get

$$k_1 = 0.64 \qquad \text{and} \qquad k_2 = -1,$$

which are the required values of $k_1$ and $k_2$.

**Question 4:** Use the simple Gaussian elimination method to find value of $\alpha$ for which the following matrix is non-invertible (singular)

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ 1 & 2 & 1 \\ 0 & \alpha & -2 \end{pmatrix},$$

then use the smallest positive integer value of $\alpha$ to get the unique solution of the linear system $A\mathbf{x} = [2, 8, -4]^T$. [5 Marks]

**Solution**. Using $m_{21} = 1$, $m_{31} = 0$ and $m_{32} = \dfrac{\alpha}{3}$, we get

$$A \equiv \begin{pmatrix} 1 & -1 & \alpha \\ 0 & 3 & 1-\alpha \\ 0 & \alpha & -2 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & \alpha \\ 0 & 3 & 1-\alpha \\ 0 & 0 & \dfrac{(\alpha^2-\alpha-6)}{3} \end{pmatrix} = U.$$

To show that the given matrix is singular, we have to show that

$$\det(A) = \det(U) = (1)(3)((\alpha^2 - \alpha - 6)/3) = (\alpha - 3)(\alpha + 2) = 0.$$

Solving the above quadratic equation, we get, $\alpha = 3$ and $\alpha = -2$, the possible values of $\alpha$ which make the given matrix singular.
To find the unique solution we take the smallest positive integer value $\alpha = 1$ and using $m_{21} = 2$, $m_{31} = 0$ and $m_{32} = \dfrac{1}{3}$, gives:

$$[A|\mathbf{b}] = \left(\begin{array}{rrr:r} 1 & -1 & 1 & 2 \\ 1 & 2 & 1 & 8 \\ 0 & 1 & -2 & -4 \end{array}\right) \equiv \left(\begin{array}{rrr:r} 1 & -1 & 1 & 2 \\ 0 & 3 & 0 & 6 \\ 0 & 1 & -2 & -4 \end{array}\right) \equiv \left(\begin{array}{rrr:r} 1 & -1 & 1 & 2 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right) = [U|c].$$

Now expressing the set in algebraic form yields

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 2 \\ & & 3x_2 & & & = & 6 \\ & & & & -2x_3 & = & -6 \end{array}$$

Using backward substitution, we get, $x_1 = 1, \ x_2 = 2, \ x_3 = 3$, the unique solution.

**Question 5:** Use LU-factorization method with Doolittle's method ($l_{ii} = 1$) to find the value of $\alpha$ such that the following linear system has infinite many solutions. Then compute these solutions. [5 Marks]

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & & & = & 1 \\ 3x_1 & + & \alpha x_2 & + & 5x_3 & = & 8 \\ & & 7x_2 & + & 3x_3 & = & 3 \end{array}$$

**Solution.** Using Simple Gauss-elimination method, we can easily find factorization of $A$ as

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & \alpha & 5 \\ 0 & 7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 7/(\alpha-3) & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & (\alpha-3) & 5 \\ 0 & 0 & 3-35/(\alpha-3) \end{pmatrix} = LU.$$

Since by one of the property of the determinant

$$\det(A) = \det(LU) = \det(L)\det(U).$$

So when using LU decomposition by Doolittle's method, then

$$\det(A) = \det(U) = \prod_{i=1}^{n} u_{ii} = (u_{11}u_{22}\cdots u_{nn}),$$

where $\det(L) = 1$ because $L$ is lower-triangular matrix and all its diagonal elements are unity. Thus the determinant of the given matrix $A$ is

$$|A| = |U| = (\alpha-3)(3-35/(\alpha-3)) = 3\alpha - 44, \quad \alpha \neq 3.$$

So $|A| = 0$, gives, $\alpha = 44/3$ and for this value of $\alpha$ we have non-trivial solutions. By solving the lower-triangular system of the form $L\mathbf{y} = [1, 8, 3]^T$ of the form

$$L\mathbf{y} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3/5 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix} = \mathbf{b},$$

we obtained the solution $\mathbf{y} = [1, 5, 0]^T$. Now solving the upper-triangular system $U\mathbf{x} = \mathbf{y}$ of the form

$$U\mathbf{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 35/3 & 5 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \mathbf{y}.$$

If we choose $x_3 = t \in R,\ t \neq 0$, then, $x_2 = 3(1-t)/7$ and $x_1 = (4+3t)/7$, then the non-trivial solutions of the given system is $\mathbf{x}^* = [(4+3t)/7, 3(1-t)/7, t]^T$.

**Question 6:** Show that the Jacobi method and Gauss-Seidel method both converge for the following linear system

$$\begin{array}{lllllll} 9x_1 & + & 2x_2 & + & 4x_3 & = & 20 \\ x_1 & + & 10x_2 & + & 4x_3 & = & 6 \\ 2x_1 & - & 4x_2 & + & 10x_3 & = & -15 \end{array}$$

Using Gauss-Seidel method, if $\mathbf{x}^{(0)} = [0, 0, 0]^T$ and $\mathbf{x}^{(1)} = [2.2222, 0.3778, -1.7933]^T$, then compute an error bound $\|x - x^{(10)}\|$ for the approximation. [5 Marks]

**Solution.** Since the given linear system is strictly diagonally dominant (SDD), therefore, both methods guaranteed converge. Or, $\|T_J\|_\infty < 1$ and $\|T_G\|_\infty < 1$.
Here we will show that the $l_\infty$-norm of Jacobi iteration matrix $T_J$ and the $l_\infty$-norm of Gauss-Seidel iteration matrix $T_G$ is less than 1, that is

$$\|T_G\|_\infty = \|T_J\|_\infty < 1.$$

The Jacobi iteration matrix $T_J$ can be obtained from the given matrix $A$ as follows

$$T_J = -D^{-1}(L+U) = -\begin{pmatrix} 1/9 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/10 \end{pmatrix} \begin{pmatrix} 0 & 2 & 4 \\ 1 & 0 & 4 \\ 2 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2/9 & -4/9 \\ -1/10 & 0 & -4/10 \\ -2/10 & 4/10 & 0 \end{pmatrix}.$$

Then the $l_\infty$ norm of the matrix $T_J$ is

$$\|T_J\|_\infty = \max\left\{\frac{6}{9}, \frac{5}{10}, \frac{6}{10}\right\} = \max\{0.6667, 0.5000, 0.6000\} = 0.6667 < 1.$$

Since the Gauss-Seidel iteration matrix is defined as

$$T_G = -(D+L)^{-1}U,$$

and by using the given information, we have

$$T_G = -\begin{pmatrix} 1/9 & 0 & 0 \\ -1/90 & 1/10 & 0 \\ -2/75 & 1/25 & 1/10 \end{pmatrix} \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2/9 & -4/9 \\ 0 & 1/45 & -16/45 \\ 0 & 4/75 & -4/75 \end{pmatrix}.$$

Then the $l_\infty$ norm of the matrix $T_G$ is

$$\|T_G\|_\infty = \max\left\{\frac{6}{9}, \frac{17}{45}, \frac{8}{75}\right\} = \max\{0.6667, 0.3778, 0.1067\} = 0.6667 < 1.$$

Using the error bound formula of Gauss-Seidel method, we obtain

$$\|\mathbf{x} - \mathbf{x}^{(10)}\| \leq \frac{(0.6667)^{10}}{1 - 0.6667} \left\| \begin{pmatrix} 2.2222 \\ 0.3778 \\ -1.7933 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\| \leq (0.0521)(2.2222) = 0.1157.$$