King Saud University: Mathematics Department Math-254
First Semester 1447 H First Midterm Exam. Solution
Maximum Marks = 25 Time: 90 mins.

Questions: (6 + 6 + 6 + 7) Marks

Question 1:

Show that Newton's iterative scheme to find the approximation to the root $\alpha = 0$ of the following nonlinear equation

$$xe^{-x} = 0,$$

is

$$x_{n+1} = x_n - \frac{x_n}{(1-x_n)}, \qquad n \geq 0.$$

Use developed iterative scheme to find the second approximation when $x_0 = 0.2$. Show that rate of convergence for the given scheme is quadratic.

Solution. Given

$$f(x) = xe^{-x} = 0, \quad \text{and} \quad f'(x) = (1-x)e^{-x}.$$

Using Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n e^{-x_n}}{(1-x_n)e^{-x_n}} = x_n - \frac{x_n}{(1-x_n)}.$$

So using developed iterative scheme by taking $x_0 = 0.2$, we get first two approximations

$$x_1 = x_0 - \frac{x_0}{(1-x_0)} = -0.0500,$$

and

$$x_2 = x_1 - \frac{x_1}{(1-x_1)} = -0.00238.$$

To check the rate of convergence, we do as

$$x_{n+1} = x_n - \frac{x_n}{(1-x_n)} = g(x_n).$$

Thus

$$\begin{aligned}
g(x) &= x - \frac{x}{(1-x)}, & g(0) = 0 \\
g'(x) &= 1 - \frac{1}{(1-x)^2}, & g'(0) = 0 \\
g''(x) &= \frac{-2}{(1-x)^3}, & g''(0) = -2 \neq 0
\end{aligned}$$

Hence the rate of convergence of given scheme is quadratic. •

Question 2:

Use secant method to find the second approximation, using $x_0 = 1$ and $x_1 = 2$, of the value of x that produces the point on the graph of $y = \dfrac{1}{x}$ that is closest to the point $(2, 1)$. Also, find the approximation of the point.

Solution. The distance between an arbitrary point $(x, 1/x)$ on the graph of $y = 1/x$ and the point $(2, 1)$ is

$$d(x) = \sqrt{(x-2)^2 + (1/x - 1)^2} = \sqrt{x^2 - 4x + 4 + 1/x^2 - 2/x + 1}.$$

Because a derivative is needed to find the critical point of d, it is easier to work with the square of this function

$$F(x) = [d(x)]^2 = x^2 - 4x + 4 + 1/x^2 - 2/x + 1,$$

whose minimum will occur at the same value of x as the minimum of $d(x)$. To minimize $F(x)$, we need x so that

$$F'(x) = 2x - 4 - 2/x^3 + 2/x^2 = 0,$$

and from this we have,

$$f(x) = 2x - 4 - 2/x^3 + 2/x^2.$$

Applying secant iterative formula to find the approximation of this equation, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(2x_n - 4 - 2/x_n^3 + 2/x_n^2)}{(2x_n - 4 - 2/x_n^3 + 2/x_n^2) - (2x_{n-1} - 4 - 2/x_{n-1}^3 + 2/x_{n-1}^2)}, \qquad n \geq 1.$$

Finding the second approximation using the initial approximations $x_0 = 1$ and $x_1 = 2$, we get

$$x_2 = x_1 - \frac{(x_1 - x_0)(2x_1 - 4 - 2/x_1^3 + 2/x_1^2)}{(2x_1 - 4 - 2/x_1^3 + 2/x_1^2) - (2x_0 - 4 - 2/x_0^3 + 2/x_0^2)} = 2 - 1/9 = 1.8889,$$

and

$$x_3 = x_2 - \frac{(x_2 - x_1)(2x_2 - 4 - 2/x_2^3 + 2/x_2^2)}{(2x_2 - 4 - 2/x_2^3 + 2/x_2^2) - (2x_1 - 4 - 2/x_1^3 + 2/x_1^2)} = 1.8667.$$

The point on the graph that is closest to the given point $(2, 1)$ has the approximate coordinates $(1.8667, 0.5356)$. •

Question 3:

Use quadratic convergence method to find the second approximation of the root $\alpha = 1$ of the following nonlinear equation

$$(x-1)^2 \sin x = 0, \quad (x \textit{ in radian}),$$

using $x_0 = 1.5$. Compute the relative error.

Solution. Since given

$$f(x) = (x-1)^2 \sin x \quad \text{and} \quad f(1) = (1-1)^2 \sin 1 = 0,$$

so $\alpha = 1$ is the root of the given nonlinear equation. To check its type, we do as

$$f'(x) = 2(x-1)\sin x + (x-1)^2 \cos x, \quad \text{and} \quad f'(1) = 2(1-1)\sin 1 + (1-1)^2 \cos 1 = 0,$$

which shows that $\alpha = 1$ is the multiple root. Thus the quadratic convergence method is modified Newton's method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n \geq 0,$$

for $n \geq 0$. The fixed point form of the developed Newton's formula is

$$x_{n+1} = g(x_n) = x_n - m\frac{(x_n - 1)\sin x_n}{(2\sin x_n + (x_n - 1)\cos x_n)},$$

where m is the order of multiplicity of the zero of the function. To find m, we check that

$$f''(x) = 2\sin x + 4(x-1)\cos x - (x-1)^2 \sin x, \quad \text{and} \quad f''(1) = 2\sin 1 \neq 0,$$

so $m = 2$. Thus

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)} = x_n - 2\frac{(x_n - 1)\sin x_n}{(2\sin x_n + (x_n - 1)\cos x_n)}, \qquad n \geq 0.$$

Now using initial approximation $x_0 = 1.5$, we have the following two approximations

$$x_1 = x_0 - 2\frac{(x_0 - 1)\sin x_0}{(2\sin x_0 + (x_0 - 1)\cos x_0)} = 1.0087, \quad x_2 = x_1 - 2\frac{(x_1 - 1)\sin x_1}{(2\sin x_1 + (x_1 - 1)\cos x_1)} = 1.0000,$$

with

$$\frac{|\alpha - x_2|}{|\alpha|} = \frac{|1 - 1.0000|}{1} = 0.0000,$$

the possible relative, up to 4 decimal places. •

<u>Question 4:</u>

(**a**) Use the simple Gaussian elimination method to find the inverse A^{-1} in α ($\neq -1$) of the following matrix

$$A = \begin{pmatrix} 1 & -1 & 2\alpha \\ 2 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix}.$$

(**b**) Find the unique solution of the system $A\mathbf{x} = [-4, 8, 12]^T$ by taking the largest possible negative integer value of α in A^{-1} got in part (**a**).

Solution. (**a**) Suppose that the inverse $A^{-1} = B$ of the given matrix exists and let

$$AB = \begin{pmatrix} 1 & -1 & 2\alpha \\ 2 & 2 & 4 \\ 0 & 4 & 8 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Now to find the elements of the matrix B, we apply the simple Gaussian elimination on the augmented matrix

$$[A|\mathbf{I}] = \left(\begin{array}{ccc|ccc} 1 & -1 & 2\alpha & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 0 & 4 & 8 & 0 & 0 & 1 \end{array}\right).$$

Using $m_{21} = \dfrac{2}{1} = 2$ and $m_{31} = 0$, gives

$$[A|\mathbf{I}] \equiv \left(\begin{array}{ccc:ccc} 1 & -1 & 2\alpha & 1 & 0 & 0 \\ 0 & 4 & 4-4\alpha & -2 & 1 & 0 \\ 0 & 4 & 8 & 0 & 0 & 1 \end{array}\right).$$

Using $m_{32} = \dfrac{4}{4} = 1$, gives

$$[A|\mathbf{I}] \equiv \left(\begin{array}{ccc:ccc} 1 & -1 & 2\alpha & 1 & 0 & 0 \\ 0 & 4 & 4-4\alpha & -2 & 1 & 0 \\ 0 & 0 & 4+4\alpha & 2 & -1 & 1 \end{array}\right) \equiv [U|C].$$

Now solve the first upper system by taking first column of matrix C, gives

$$\begin{pmatrix} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix},$$

by using backward substitution, we get

$$\begin{array}{rcrcrcr} b_{11} & - & b_{21} & + & 2\alpha b_{31} & = & 1 \\ & & 4b_{21} & + & (4-4\alpha)b_{31} & = & -2 \\ & & & + & (4+4\alpha)b_{31} & = & 2 \end{array}$$

which gives $b_{11} = 0,\ b_{21} = -\frac{1}{1+\alpha},\ b_{31} = \frac{1}{2(1+\alpha)}$. Similarly, the solution of the second upper system by taking second column of matrix C, gives

$$\begin{pmatrix} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcrcr} b_{12} & - & b_{22} & + & 2\alpha b_{32} & = & 0 \\ & & 4b_{22} & + & (4-4\alpha)b_{32} & = & 1 \\ & & & + & (4+4\alpha)b_{32} & = & -1 \end{array}$$

which gives $b_{12} = \frac{1}{2},\ b_{22} = \frac{1}{2(\alpha+1)},\ b_{32} = -\frac{1}{4(\alpha+1)}$. Finally, the solution of the third upper system by taking third column of matrix C, gives

$$\begin{pmatrix} 1 & -1 & 2\alpha \\ 0 & 4 & 4-4\alpha \\ 0 & 0 & 4+4\alpha \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcrcr} b_{13} & - & b_{23} & + & 2\alpha b_{33} & = & 0 \\ & & 4b_{23} & + & (4-4\alpha)b_{33} & = & 0 \\ & & & + & (4+4\alpha)b_{33} & = & 1 \end{array}$$

and it gives $b_{13} = -\frac{1}{4}$, $b_{23} = \frac{\alpha-1}{4(\alpha+1)}$, $b_{33} = \frac{1}{4(\alpha+1)}$. Hence the elements of the inverse matrix B are Thus

$$B = A^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{4} \\ \\ -\frac{1}{1+\alpha} & \frac{1}{2(\alpha+1)} & \frac{\alpha-1}{4(\alpha+1)} \\ \\ \frac{1}{2(1+\alpha)} & -\frac{1}{4(\alpha+1)} & \frac{1}{4(\alpha+1)} \end{pmatrix},$$

which is the required inverse of the given matrix A for any α.
(**b**) The inverse does not exist at $\alpha = -1$, so taking the largest possible negative integer value of α in A^{-1} which will be $\alpha = -2$, so we get

$$B = A^{-1} = \begin{pmatrix} 0 & 0.5 & -0.25 \\ 1 & -0.5 & 0.75 \\ -0.5 & 0.25 & -0.25 \end{pmatrix},$$

which is the required inverse of the given matrix A.
Now to find the solution of the system, we do as

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 0 & 0.5 & -0.25 \\ 1 & -0.5 & 0.75 \\ -0.5 & 0.25 & -0.25 \end{pmatrix} \begin{pmatrix} -4 \\ 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the required solution. •