**King Saud University: Mathematics Department Math-254**
**First Semester 1445 H First Midterm Exam.**
**Solution**
**Maximum Marks = 25 Time: 90 mins.**

---

**Question 1:** Show that $\alpha = 1$ is root of the nonlinear equation $1 = xe^{1-x}$. Use the best numerical method to find the second approximation $x_2$ to this root using initial approximation $x_0 = 0.75$. Compute the absolute error. [6 Marks]

**Solution.** To check $\alpha = 1$ is a root of $1 - xe^{1-x} = 0$, we do

$$f(x) = 1 - xe^{1-x}, \quad f(1) = 1 - 1e^{1-1} = 1 - 1 = 0,$$

which shows that $\alpha = 1$ is root of the given equation. Now we check the type of the root as

$$f'(x) = -e^{1-x} + xe^{1-x} = (x-1)e^{1-x}, \quad f'(1) = 0,$$

which shows that the $\alpha = 1$ is a multiple root of the given equation. To find its order of multiplicity, we do as

$$f''(x) = e^{1-x} - (x-1)e^{1-x} = (2-x)e^{1-x}, \quad f''(1) = 1 \neq 0,$$

which shows that the order of multiplicity of the multiple root is 2. So using modified Newton's method by taking $x_0 = 0.75$, we get first two approximation

$$x_1 = x_0 - 2\frac{1 - x_0e^{1-x_0}}{(x_0-1)e^{1-x_0}} = 0.9804 \quad \text{and} \quad x_2 = x_1 - 2\frac{1 - x_0e^{1-x_0}}{(x_0-1)e^{1-x_0}} = 0.9999,$$

and the absolute error is, $|\alpha - x_2| = |1 - 0.9999| = 0.0001$. •

**Question 2:** The nonlinear equation $f(x) = \tan x = 0$ has a simple root $\alpha = \pi$. Show that the Newton's method for approximating this root is,

$$x_{n+1} = x_n - \sin(x_n)\cos(x_n), \quad n \geq 0.$$

Then use it to find the second approximation $x_2$ using initial approximation $x_0 = 3.0$. Find the rate of convergence of the developed formula. [7 Marks]

**Solution.** As $f(x) = \tan x$ and so $f'(x) = \sec^2 x$, and

$$f(\pi) = \tan(\pi) = 0, \qquad f'(\pi) = \sec^2(\pi) \neq 0,$$

therefore, the root is the simple root of the given nonlinear equation and the Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan(x_n)}{\sec^2(x_n)},$$

or

$$x_{n+1} = x_n - \frac{\sin(x_n)\cos^2(x_n)}{\cos(x_n)} = x_n - \sin(x_n)\cos(x_n), \ n \geq 0.$$

To find the second approximation to the root by using above scheme using $x_0 = 3.0$, we obtain

$$x_1 = x_0 - \sin(x_0)\cos(x_0) = 3.1397, \quad x_2 = x_1 - \sin(x_1)\cos(x_1) = 3.1416,$$

which gives absolute error

$$|\pi - x_2| = |3.1416 - 3.1416| = 0, \quad 4 \ dp.$$

Since the fixed-point form of the Newton's method for the given problem is,

$$g(x) = x - \sin(x)\cos(x) = x - \frac{1}{2}\sin(2x),$$

therefore,

$$\begin{aligned}
g(x) &= x - \sin(x)\cos(x), \quad g(\pi) = \pi - \sin(\pi)\cos(\pi) = \pi, \\
g'(x) &= 1 - \cos^2(x) + \sin^2(x) = 0, \quad g'(\pi) = 1 - \cos^2(\pi) + \sin^2(\pi) = 1 - 1 + 0 = 0, \\
g''(x) &= 2\cos(x)\sin(x) + 2\sin(x)\cos(x) = 4\sin(x)\cos(x), \quad g''(\pi) = 4\sin(\pi)\cos(\pi) = 0, \\
g'''(x) &= 4\cos^2(x) - 4\sin^2(x), \quad g'''(\pi) = 4\cos^2(\pi) - 4\sin^2(\pi) = 4(1) - 4(0) = 4 \neq 0.
\end{aligned}$$

Hence the convergence of the Newton's method is cubic. •

**Question 3:** Successive approximations $x_n$ to the desired root are generated by the scheme

$$x_{n+1} = \frac{e^{x_n}(x_n + 1) + 2x_n^2}{e^{x_n} + 3x_n}, \qquad n \geq 0.$$

Find the nonlinear equation $f(x) = 0$. Use the secant method to find the second approximation $x_3$ of the root $\alpha = -0.7035$, starting with initial approximations $x_0 = -0.5$ and $x_1 = -0.25$. Compute the relative error. [6 Marks]

**Solution.** Given

$$x_{n+1} = \frac{e^{x_n}(x_n + 1) + 2x_n^2}{e^{x_n} + 3x_n} = g(x_n), \qquad n \geq 1.$$

$$x = \frac{e^{x}(x + 1) + 2x^2}{e^{x} + 3x} = g(x),$$

$$g(x) - x = \frac{e^{x}(x + 1) + 2x^2}{e^{x} + 3x} - x = 0,$$

$$g(x) - x = \frac{e^{x}(x + 1) + 2x^2 - x(e^x + 3x)}{e^{x} + 3x} = 0,$$

and after simplifying, we obtained

$$g(x) - x = \frac{(xe^x + e^x + 2x^2 - xe^x - 3x^2)}{e^x + 3x} = \frac{(e^x - x^2)}{e^x + 3x} = -\frac{(x^2 - e^x)}{e^x + 3x} = x^2 - e^x = 0.$$

Thus

$$f(x) = g(x) - x = x^2 - e^x = 0.$$

Applying secant iterative formula to find the approximation of this zero, we use the formula

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})(x_n^2 - e^{x_n})}{(x_n^2 - e^{x_n}) - (x_{n-1}^2 - e^{x_{n-1}})}, \qquad n \geq 1.$$

Finding the third approximation using the initial approximations $x_0 = -0.5$ and $x_1 = -0.25$, we get

$$x_2 = x_1 - \frac{(x_1 - x_0)(x_1^2 - e^{x_1})}{(x_1^2 - e^{x_1}) - (x_0^2 - e^{x_0})} = -0.7477,$$

$$x_3 = x_2 - \frac{(x_2 - x_1)(x_2^2 - e^{x_2})}{(x_2^2 - e^{x_2}) - (x_1^2 - e^{x_1})} = -0.6946,$$

and the relative error is, $\dfrac{|\alpha - x_3|}{|\alpha|} = \dfrac{|-0.7035 - (-0.6946)|}{|-0.7035|} = 0.0127.$ •

**Question 4:** Use Simple Gauss-elimination method to find the many solutions of the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ -1 & 2 & -\alpha \\ \alpha & 1 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

by using the suitable value of the $\alpha$. [6 Marks]

**Solution.** Using $m_{21} = -1$, $m_{31} = \alpha$ and $m_{32} = \dfrac{1+\alpha}{1}$, gives matrix form

$$[A|b] = \begin{pmatrix} 1 & -1 & \alpha & 1 \\ -1 & 2 & -\alpha & 1 \\ \alpha & 1 & 1 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & \alpha & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1+\alpha & 1-\alpha^2 & -1-\alpha \end{pmatrix} \equiv \begin{pmatrix} 1 & -1 & \alpha & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1-\alpha^2 & -3-3\alpha \end{pmatrix}.$$

Obviously, the original set of equations has been transformed to an upper-triangular form.
So if $1 - \alpha^2 \neq 0$, then we have the unique solution of the given system while for $\alpha = \pm 1$, we have no unique solution. If $\alpha = -1$, then we have infinitely many solution because third row of above augmented matrix gives

$$0x_1 + 0x_2 + 0x_3 = 0,$$

and when $\alpha = 1$, we have

$$0x_1 + 0x_2 + 0x_3 = -6,$$

which is not possible, so no solution.
Thus taking suitable value of $\alpha = -1$, we have upper-triangular system of the form

$$\begin{array}{ccccccc} x_1 & - & x_2 & - & x_3 & = & 1 \\ & & x_2 & & & = & 2 \end{array}$$

Performing backward substitution and using $x_3 = t \in R,\ t \neq 0$, yields, $x_2 = 2$ and $x_1 = 3 + t$, the required many solutions of the given system using $\alpha = -1$. •