King Saud University
College of Sciences
Department of Mathematics

First Midterm Exam, S2 1446
M 507 - Advanced Operations Research
Time: 2 hours – **Marks: 30**

Answer the following questions.

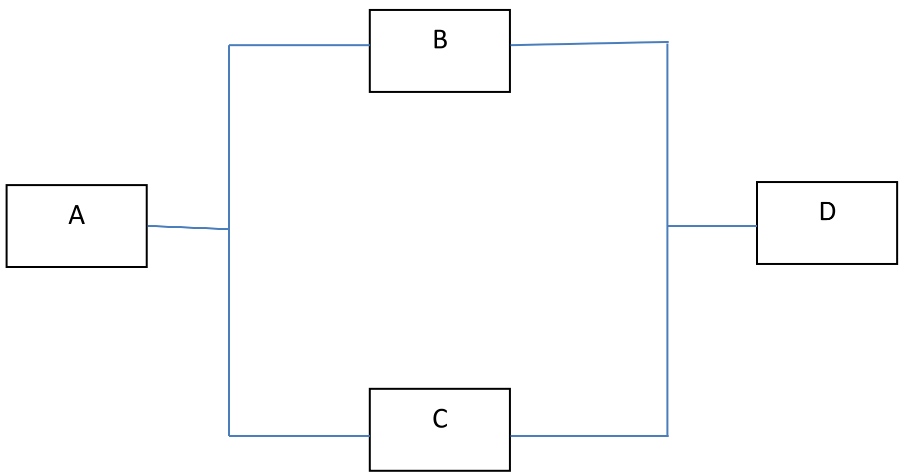
Q1: [4+2+4]

In the reliability diagram below, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

(a) Decomposition using B as the keystone element.

(b) The reduction method.

(c) Compute the importance of each component if $R_A = 0.8$, $R_B = 0.9$, $R_C = 0.95$ and $R_D = 0.98$



Q2: [5+5]

(a) A machine with constant failure rate λ will survive a period of 100 hours without failure, with probability 0.50.

(i) Determine the failure rate λ

(ii) Find the probability that the machine will survive 500 hours without failure.

(iii) Determine the probability that the machine will fail within 1000 hours, when you know that the machine was functioning at 500 hours.

(b) Let $T|\Lambda$ have an exponential distribution with parameter λ. Let Λ have a gamma distribution with parameters k and α. Determine the probability density function for the unconditional distribution of T, then find the survivor function $R(t)$, the failure rate function $Z(t)$, and the MTTF.

Hint: If $X \sim gamma(k,\alpha)$ then $f(x) = \dfrac{\alpha^k x^{k-1} e^{-\alpha x}}{\Gamma(k)} \quad x>0,\ \alpha > 0,\ k > 0$

Q3: [3+3+4]

(a) For the Markov process $\{X_t\},\ t = 0,1,2,\ldots,n$ with states $i_0, i_1, i_2,\ \ldots\ , i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2,\ \ldots\ , X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$

(b) If a Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5,\ p_1 = 0.2$ and $p_2 = 0.3$ Find $pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$

(c) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "defective" and $X_n = 1$ meaning "good". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.89 & 0.11 \\ 0.02 & 0.98 \end{array} \right\| \end{array}$$

What is the probability that the fourth item is defective given that the first item is good?

Model Answer

Q1: [4+2+4]

In the reliability diagram shown in Fig. 1, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

a) Decomposition using B as the keystone element.

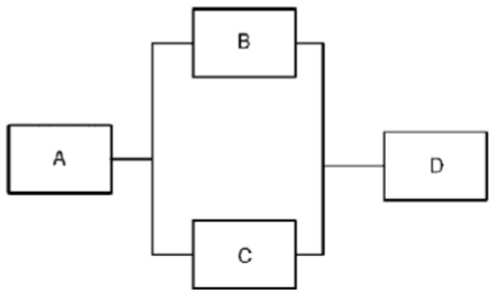


Fig. 1: Reliability diagram

Using B as the keystone element, we have two cases i.e., the case when B functions and the case when it does not.

For the case when B functions, the system reduced to Fig 2.

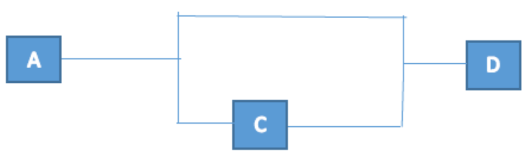


Fig. 2: The case when B functions

Thus the reliability of the system depends only on the reliability of component A and D. Note that $R_A = R_B = R_C = R_D = R$

Therefore,

$$R^+ = R_A R_D = R^2$$

For the case when B fails, the system block is as shown in Fig. 3, which is a series system.

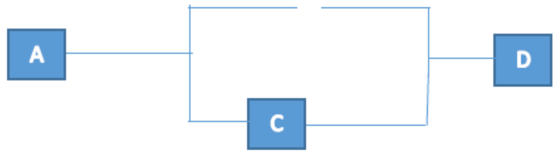


Fig. 3: The case when B fails to work

Thus the reliability of the system depends on A, C, and D, therefore we have:

$$R^{-} = R_A R_C R_D = R^3$$

Thus the reliability of the system using the two decompositions is given as:

$$R_{system} = R_B R^{+} + (1 - R_B) R^{-}$$
$$R_{system} = R(R^2) + (1 - R) R^3$$
$$R_{system} = 2R^3 - R^4$$

b) Using the reduction method

With this method, it can be seen that components B and C are in parallel and jointly in series with A and D. therefore the reduced system is given in Fig. 4.

Fig. 4: Reduced system

For parallel components B and C, we have

$$R_{B\|C} = 1 - \prod_{i=1}^{2}(1 - R_i)$$
$$R_{B\|C} = R_B + R_C - R_B R_C$$
$$R_{B\|C} = 2R - R^2$$

The reliability of the system is thus given as:

$$R_{system} = R_A R_{B\|C} R_D$$
$$R_{system} = R(2R - R^2)R$$
$$R_{system} = 2R^3 - R^4$$

c)

Recall that the reliability of the system is given as:

$$R_{system} = R_A R_D (R_B + R_C - R_B R_C)$$

The importance of each component is computed by taking the partial derivative with respect to each of the component.

Thus the importance of component A is given as:

$$\frac{\delta R_{system}}{\delta R_A} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_A}$$
$$I_A = R_D (R_B + R_C - R_B R_C)$$

$$\Rightarrow \mathrm{I_A} = 0.98(0.9 + 0.95 - 0.9 \times 0.95)$$
$$= 0.9751$$

The importance of component B is given as:

$$\frac{\delta R_{system}}{\delta R_B} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_B}$$
$$I_B = R_A R_D - R_A R_D R_C$$

$$\Rightarrow \mathrm{I_B} = 0.8(0.98) - 0.8(0.98)(0.95)$$
$$= 0.0392$$

The importance of component C is given as:

$$\frac{\delta R_{system}}{\delta R_C} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_C}$$
$$I_C = R_A R_D - R_A R_B R_D$$

$$\Rightarrow \mathrm{I_c} = 0.8(0.98) - 0.8(0.9)(0.98)$$
$$= \ 0.0784$$

The importance of component D is given as:

$$\frac{\delta R_{system}}{\delta R_D} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_D}$$
$$I_D = R_A (R_B + R_C - R_B R_C)$$

$$\Rightarrow \mathrm{I_D} = 0.8(0.9 + 0.95 - 0.9 \times 0.95)$$
$$= 0.796$$

Q2: [5+5]

(a)
Since the failure rate is constant, therefore the time to failure follows an exponential distribution.

The PDF is $f(t) = \lambda e^{-\lambda t} \quad t > 0,\ \lambda > 0$

The reliability function is $R(t) = e^{-\lambda t}$

(i) To determine λ, we have $0.5 = e^{-100\lambda}$

$$\therefore \lambda = -\frac{\ln(0.5)}{100} \simeq 0.0069$$

(ii)
The probability that the machine will survive 500 hours without failure is

$$R(t) = e^{-500(0.0069)} \simeq 0.03$$

(iii)

$$\because R(x|t) = R(x)$$

$i.e. \Pr(T > t + x | T > t) = \Pr(T > x)$ memoryless property for exp. distribution.

The survival function of an item that has been functioning for t time units is therefore equal to the survival function of a new item.

$i.e.\ R(x|t) = R(x)$

Also, $\Pr(T \le t + x | T > t) = \Pr(T \le x)$

$$\therefore \Pr(T \le 1000 | T > 500) = \Pr(T \le 500)$$

$$= 1 - e^{-500(0.0069)} \simeq 0.97$$

which is the probability that the machine will fail within 1000 hours, knowing that the machine was functioning at 500 hours.

(b)

$$f(t|\lambda) = \lambda e^{-\lambda t} \quad t > 0 \quad (1)$$

$$\because \Lambda \sim gamma(k, \alpha)$$

$$\therefore \pi(\lambda) = \frac{\alpha^k \lambda^{k-1} e^{-\alpha\lambda}}{\Gamma(k)} \quad \lambda > 0,\ \alpha > 0,\ k > 0 \quad (2)$$

The unconditional or mixture or marginal density function of T is given by

$$f(t) = \int_0^\infty f(t|\lambda)\pi(\lambda)d\lambda \quad (3)$$

$$f(t) = \frac{\alpha^k}{\Gamma(k)} \int_0^\infty \lambda^k e^{-(\alpha+t)\lambda} d\lambda$$

$$= \frac{\alpha^k}{\Gamma(k)(\alpha+t)^{k+1}} \int_0^\infty u^k e^{-u} du, \quad u = (\alpha + t)\lambda$$

$$\therefore\ f(t) = \frac{\Gamma(k+1)\alpha^k}{\Gamma(k)(\alpha+t)^{k+1}} = \frac{k\alpha^k}{(\alpha+t)^{k+1}} \qquad (4)$$

The survivor function is

$$R(t) = \Pr(T > t) = \int_t^\infty f(u)du$$

$$= \int_t^\infty \frac{k\alpha^k}{(\alpha+u)^{k+1}} du = k\alpha^k \int_t^\infty (\alpha+u)^{-k-1} du$$

$$\therefore R(t) = -\frac{\alpha^k}{(\alpha+u)^k}\bigg|_t^\infty = \left(\frac{\alpha}{\alpha+t}\right)^k = \left(1+\frac{t}{\alpha}\right)^{-k} \qquad (5)$$

The mean time to failure is

$$MTTF = \int_0^\infty R(t)dt$$

$$= \int_0^\infty \left(1+\frac{t}{\alpha}\right)^{-k} dt$$

$$= -\alpha \frac{(1+t/\alpha)^{1-k}}{k-1}\bigg|_0^\infty$$

$$\therefore MTTF = \frac{\alpha}{k-1} \quad k > 1 \qquad (6)$$

Note that $MTTF$ does not exist for $0 \le k \le 1$.

The failure rate function is $Z(t) = \dfrac{f(t)}{R(t)} = \dfrac{k}{\alpha+t}$ which is a decreasing function with t.

Q3: [3+3+4]

(a)

$$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$$

$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}$$

$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n} \quad \text{Definition of Markov}$$

By repeating this argument $n-1$ times

$$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_n = i_n\}$$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \dots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

$\mathrm{pr}\left\{X_1=1, X_2=1, X_3=0\right\}=p_1 P_{11} P_{10}$, $\quad p_1=\mathrm{pr}\left\{X_1=1\right\}$

$\mathrm{pr}\left\{X_1=1\right\}=\Pr(X_1=1|X_0=0)\Pr(X_0=0)+\Pr(X_1=1|X_0=1)\Pr(X_0=1)+\Pr(X_1=1|X_0=2)\Pr(X_0=2)$

$=P_{01}p_0+P_{11}p_1+P_{21}p_2$

$=0.3(0.5)+0.2(0.2)+0.3(0.3)=0.28$

$\therefore\ pr\left\{X_1=1, X_2=1, X_3=0\right\}=\ 0.28(0.2)(0.4)=0.0224$

(c)

$$P^2=\begin{bmatrix}0.89 & 0.11\\ 0.02 & 0.98\end{bmatrix}\begin{bmatrix}0.89 & 0.11\\ 0.02 & 0.98\end{bmatrix}$$

$$P^2=\begin{bmatrix}0.7943 & 0.2057\\ 0.0374 & 0.9626\end{bmatrix}$$

$$P^3=\begin{bmatrix}0.7110 & 0.2890\\ 0.0525 & 0.9475\end{bmatrix}$$

$pr\left\{X_3=0|X_0=1\right\}=p_{10}^3=0.0525$

$= 5.25\%$,

which is the probability that the fourth item is defective given that the first item is good.