King Saud University
College of Sciences
Department of Mathematics

Mid-Term Exam, S2 1444
M 507 - Advanced Operations Research
Time: 2 hours

Answer the following questions.

Q1: [3+3+2]

Three components, a, b, and c have reliabilities 0.99, 0.95, and 0.90, respectively.

Which of the following two options results in a higher system reliability?

(i) The system consists of a parallel arrangement of two of modules where each module includes components a, b, and c in series.

(ii) The system consists of a parallel combination of two components of type a in series with similar parallel combinations of b and c. This is called low-level redundancy.

If in the low-level redundancy arrangement, it was possible to add a third component of type a, then determine the reliability for this arrangement.

Q2: [4+4]

(a) Let $T > 0$ be a random variable, its hazard function is $h(t)$.

Prove each of the following:

(i) The reliability function is given by $R(t) = \exp\left\{-\int_0^t h(u)du\right\}$, then determine $R(t)$ if $T \sim \exp(\lambda)$.

(ii) The average failure rate over the interval (t_1, t_2) is $\bar{\lambda} = \frac{\ln[R(t_1)] - \ln[R(t_2)]}{t_2 - t_1}$, then derive a formula for $\bar{\lambda}$ if T follows a two-parameter Weibull distribution.

(b) Derive formulas to find the mode, the p-th percentile and the median of the two parameter Weibull distribution.

Q3: [2+2+4]

The life of a product follows a Weibull distribution with a shape parameter of 3.5 and a scale parameter of 800 hours. Find each of the following:

(i) The probability that the product will perform satisfactory for at least 100 hours?

(ii) Compute the instantaneous failure rate at its characteristic value and the average failure rate over the time interval $(400, 600)$.

(iii) Compute the mode, the tenth percentile, the median, the MTTF and the variance for this distribution.

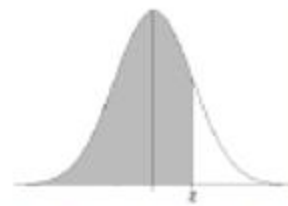
Q4: [3+3]

(a) Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10000 miles. If a person desires to take a 9000-mile trip, what is the probability that he will be able to complete his trip without having to replace the car battery?

(b) A fraction $p = 0.05$ of the items coming off of a production process are defective. The output of the process is sampled, one by one, in a random manner. What is the probability that the first defective item found is the tenth item sampled?

Table 1

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 2

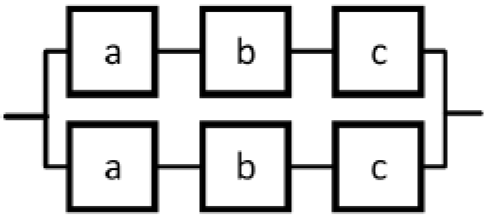
$B_2 - B_1^2 = \Gamma(2/\beta + 1) - \Gamma^2(1/\beta + 1)$ as a Function of the Shape Parameter β

β	B_1	$B_2 - B_1^2$
1.0	1.0000	1.0000
1.1	0.9649	0.7714
1.2	0.9407	0.6197
1.3	0.9336	0.5133
1.4	0.9114	0.4351
1.5	0.9027	0.3757
1.6	0.8966	0.3292
1.7	0.8922	0.2919
1.8	0.8893	0.2614
1.9	0.8874	0.2360
2.0	0.8862	0.2146
2.5	0.8873	0.1441
3.0	0.8930	0.1053
3.5	0.8997	0.0811
4.0	0.9064	0.0647
5.0	0.9182	0.0442

Model Answer

Q1: [3+3+2]

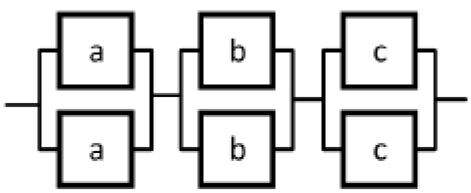
(i) The high-level redundancy system is shown as in the opposite Fig.



In this system, the reliability can be calculated as follows:

$$R_{sys} = R_a R_b R_c + R_a R_b R_c - (R_a R_b R_c)(R_a R_b R_c)$$
$$= 2R_a R_b R_c - (R_a R_b R_c)^2$$
$$= 2(0.99)(0.95)(0.9) - [(0.99)(0.95)(0.9)]^2$$
$$\therefore R_{sys} = 0.9764$$

(ii) The low-level redundancy system is shown as in the opposite Fig.

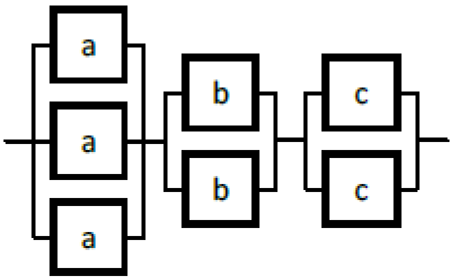


In this system, the reliability can be calculated as follows:

$$R_{sys} = (2R_a - R_a{}^2)(2R_b - R_b{}^2)(2R_c - R_c{}^2)$$
$$= (2\times 0.99 - 0.99^2)(2\times 0.95 - 0.95^2)(2\times 0.9 - 0.9^2)$$
$$\therefore R_{sys} = 0.9874$$

So, the low-level redundancy system gives higher reliability.

After adding a third component of type a in the low-level redundancy arrangement, as shown in the opposite Fig.



$$R_{sys} = [1-(1-R_a)^3][1-(1-R_b)^2][1-(1-R_c)^2]$$
$$= (1-0.01^3)(1-0.05^2)(1-0.1^2)$$
$$\therefore R_{sys} = 0.9875$$

Q2: [4+4]

(a)

(i)

To prove that $R(t) = \exp\left\{-\int_0^t h(u)du\right\}$

The hazard function or failure rate is given by $h(t) = \dfrac{f_T(T)}{R(t)}$

$$\Rightarrow \begin{array}{l} h(t) = \dfrac{d}{dt}F_T(t).\dfrac{1}{R(t)} \\ h(t) = -\dfrac{d}{dt}R(t).\dfrac{1}{R(t)} \end{array}$$

$$\Rightarrow \int_0^t \frac{dR(u)}{R(u)} = -\int_0^t h(u)du$$

$$\therefore [\ln R(u)]_0^t = -\int_0^t h(u)du$$

$$\therefore \ \ln R(t) - \ln R(0) = -\int_0^t h(u)du \ , \ \because \ \ln R(0) = \ln(1) = 0$$

$$\therefore \ \ln R(t) = -\int_0^t h(u)du$$

$$\therefore \ R(t) = e^{-\int_0^t h(u)du}$$

$$\therefore R(t) = \exp\left\{-\int_0^t h(u)du\right\}$$

$$\because T \sim \exp(\lambda)$$

$$\therefore R(t) = \exp\left\{-\int_0^t \lambda du\right\}$$

$$= \exp(-\lambda t)$$

(ii)

$$\because \ R(t) = e^{-\int_0^t h(u)du}$$

$$\therefore \ R(t) = e^{-\Lambda(t)}, \ \Lambda(t) = \int_0^t h(u)du$$

The average failure rate over the interval (t_1, t_2) is given by

$$\overline{\lambda} = \frac{\int_{t_1}^{t_2} h(u)du}{t_2 - t_1}$$

$$\therefore \ \overline{\lambda} = \frac{\int_0^{t_2} h(u)du - \int_0^{t_1} h(u)du}{t_2 - t_1} = \frac{\Lambda(t_2) - \Lambda(t_1)}{t_2 - t_1} \qquad (1)$$

$$\therefore \ \overline{\lambda} = \frac{\ln[R(t_1)] - \ln[R(t_2)]}{t_2 - t_1} \qquad (2)$$

For 2p Weibull

$R(t)=\exp[-(\frac{t}{\eta})^{\beta}],\; R(t)=e^{-\Lambda(t)}$

$\therefore\; \Lambda(t)=(\frac{t}{\eta})^{\beta}$ (3)

$\therefore\; (1), (3) \Rightarrow\quad \bar{\lambda}=\dfrac{t_2^{\,\beta}-t_1^{\,\beta}}{\eta^{\beta}(t_2-t_1)}$

(b)

The pdf of Weibull distribution with 2-parameters β and η is given by

$$f(x)=\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]$$

The mode is the value of x at which the pdf is largest.

The mode of Weibull distribution occurs at $x=0$ if $\boldsymbol{\beta}\le 1$

For $\beta>1,$ we should set $\dfrac{\partial f}{\partial x}=0$ to get the mode

$$\frac{\partial f}{\partial x}=\frac{\beta}{\eta}\frac{(\beta-1)x^{\beta-2}}{\eta^{\beta-1}}\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]$$

$$+\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}\left(\frac{-\beta}{\eta^{\beta}}\right)x^{\beta-1}\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]=0$$

Simplify and dividing by $x^{\beta-2}$ we get

$$\beta-1=\frac{\beta}{\eta^{\beta}}x^{\beta}$$

$$\therefore\; x_{\text{mode}}=\left[\eta^{\beta}\frac{\beta-1}{\beta}\right]^{\frac{1}{\beta}}$$

$$=\eta\left(\frac{\beta-1}{\beta}\right)^{\frac{1}{\beta}}$$

To get the p-th percentile of 2-parameter Weibull distribution, solve the equation

$F(x_p)=p,$ where $F(x)$ is the cdf of 2-parameter Weibull distribution.

$$1-\exp\left[-\left(\frac{x_p}{\eta}\right)^{\beta}\right]=p$$

$$\Rightarrow \exp\left[-\left(\frac{x_p}{\eta}\right)^{\beta}\right]=1-p$$

$$-\left(\frac{x_p}{\eta}\right)^{\beta}=\ln(1-p)$$

$$\Rightarrow \ (x_p)^{\beta}=\eta^{\beta}\ln(1-p)^{-1}$$

$$(x_p)^{\beta}= \ \eta^{\beta}\ln\left(\frac{1}{1-p}\right)$$

$$\therefore \ x_p=\left\{\ln\left(\frac{1}{1-p}\right)\right\}^{1/\beta}\eta$$

The tenth percentile for the product's life is given by

$$\mathrm{x}_{0.10}=\left(\ln\left(\frac{1}{1-0.10}\right)\right)^{1/\beta}\eta$$

$$=\left(-\ln 0.9\right)^{1/\beta}\eta$$

Also, the median is given by

$$\mathrm{x}_{0.50}=\left(\ln\left(\frac{1}{1-0.50}\right)\right)^{1/\beta}\eta$$

$$=\left(\ln 2\right)^{1/\beta}\eta$$

Q3: [2+2+4]

(i)

$$\Pr(T\ >100)=R(100)$$

$$=\exp[-(\tfrac{t}{\eta})^{\beta}]$$

$$=e^{-(\frac{100}{800})^{3.5}}$$

$$=0.9993$$

(ii)

The instantaneous failure rate is given by

$$\lambda(t) = \frac{\beta}{\eta^{\beta}} t^{\beta - 1}$$

$$\begin{aligned}
\lambda(800) &= \frac{3.5}{800^{2.5}} \times 800^{1.5} = \frac{3.5}{800} \\
&= 0.004375 \\
&= 4.375 \times 10^{-3}
\end{aligned}$$

and the average failure rate is

$$\begin{aligned}
\bar{\lambda} &= \frac{t_2^{\ \beta} - t_1^{\ \beta}}{\eta^{\beta}(t_2 - t_1)} \\
&= \frac{600^{3.5} - 400^{3.5}}{800^{3.5}(600 - 400)} \\
&= 1.3848 \times 10^{-3}
\end{aligned}$$

(iii) The Mode is given by

$$\begin{aligned}
\mathrm{x}_m &= \eta \left(\frac{\beta - 1}{\beta} \right)^{1/\beta} \\
&= 800 \left(\frac{2.5}{3.5} \right)^{1/3.5} \\
&\simeq 726.67
\end{aligned}$$

The tenth percentile is

$$\begin{aligned}
\mathrm{x}_{0.10} &= \left(\ln\left(\frac{1}{1-p} \right) \right)^{1/\beta} . \eta \\
&= \left(\ln\left(\frac{1}{0.9} \right) \right)^{1/3.5} \times 800 \\
&\simeq 420.59
\end{aligned}$$

Also, the median is

$$\begin{aligned}
\mathrm{x}_{0.50} &= \left(\ln\left(\frac{1}{1-0.50} \right) \right)^{1/3.5} \times 800 \\
&= (\ln 2)^{2/7} \times 800 \\
&\simeq 720.46
\end{aligned}$$

The MTTF for 2p Weibull is given by

$$\mu = \eta\ \Gamma\left(\frac{1}{\beta}+1\right)$$
$$= \eta\ B_1$$
$$= 800 \times 0.8997$$
$$= 719.76$$

The variance is

$$\sigma^2 = \eta^2 [B_2 - B_1^2]$$
$$= 800^2 \times 0.0811$$
$$= 51{,}904$$

Q4: [3+3]

(a)

$$\because\ X \sim\ \exp(\tfrac{1}{10000})$$
$$\therefore\ \Pr(X > 9000) = e^{-0.9}$$
$$\approx 0.4066$$

(b)

$$\because X \sim\ \text{geom}\ (0.05)$$

$$\therefore \Pr(X = k) = p(1 - p)^{k-1},\ \ k = 1, 2,\ \ldots$$

$$\therefore \Pr(X = 10) = 0.05(0.95)^9 = 0.0315$$