King Saud University
College of Sciences
Department of Mathematics

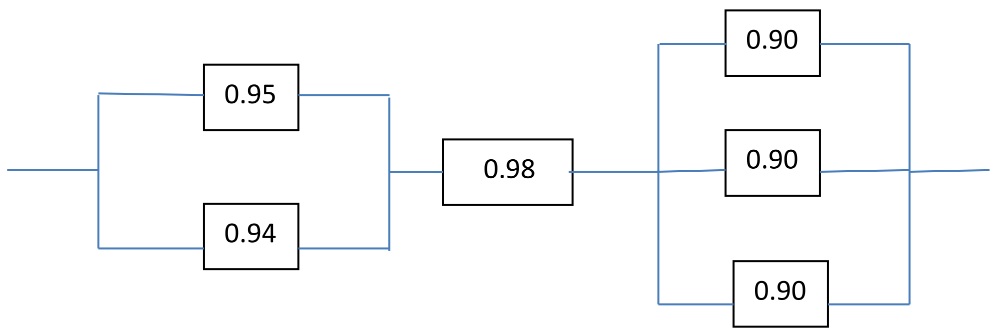
Final Exam, S2 1446
M 507 - Advanced Operations Research
Time: 3 hours – **Marks: 40**

Answer the following questions.

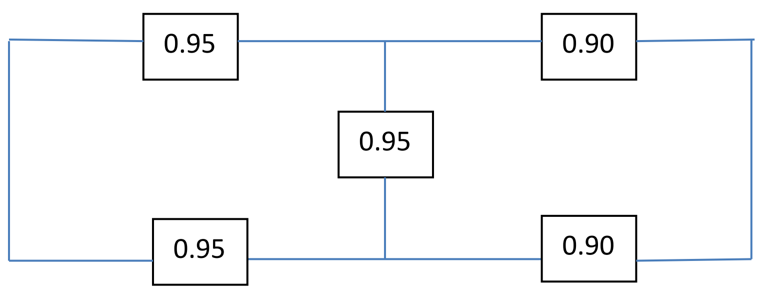
Q1: [4+4]

Compute the system reliability for the following configuration diagram where each component has the indicated reliability

(a)



(b)



Q2: [4+4]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \;\; n = 1, 2, 3, \; \ldots \qquad (2)$$

where all birth parameters are the same constant λ with initial condition $X(0)=0$, Show that $p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$, $n = 0,1,2,...$

(b) Messages arrive at a telegraph office as a Poisson Process with mean rate of 3 messages per hour.

(i) What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

(ii) What is the distribution of the time at which the first afternoon message arrives?

Q3: [4+4]

(a) In the reactor safety study, the failure rate random variable is Λ, $\Lambda \sim \text{lognormal}(\upsilon, \tau^2)$ with median $\lambda_m = 6\times10^{-5}$ and a 90% error factor $k = 3$. Find the mean failure rate.

(b) A rotary pump has a constant failure rate $\lambda = 4.28\times10^{-4}$ hours^{-1}.

(i) Determine the mean time to failure (MTTF) of the pump and find the probability that it survives the first month ($t_1 = 730$ hours).

(ii) Suppose that the pump has been functioning without failure during its first month, find the probability that the pump will fail during the next month ($t_2 = 730$ hours).

Q4: [4+4]

(a) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1-\alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(ii) Determine the probability that a correct signal is received at stage 2.

(b) Let $\{X_n,\ n \ge 0\}$ is a discrete-time Markov chain that describes evolution of a machine in a certain company with state space $\{0,1\}$, with $X_n = 0$ meaning " the machine is down at the beginning of day n" and $X_n = 1$ meaning " the machine is up at the beginning of day n". The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{Vmatrix} 0.03 & 0.97 \\ 0.02 & 0.98 \end{Vmatrix} \end{array}$$

Suppose the company maintains a system of **two** such machines that are identical and independently of each other. Let Y_n be the number of machines in the "up" state at the beginning of day n. Determine the transition probability matrix for the Markov chain $\{Y_n,\ n \geq 0\}$ with state space $\{0,1,2\}$.

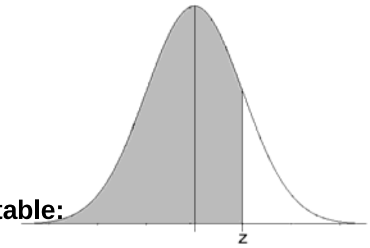
Q5: [4+4]

(a) A company produces two types of mobile phones, Model A and Model B and that it takes 5 hours to produce a unit of Model A and two hours to produce a unit of Model B, knowing that the number of working hours is 900 hours per week. Consider that the unit cost of Model A is \$8, and the unit cost of Model B is \$10 and the budget for production per week is \$2800. If the profit per unit of Model A is \$3 and the profit per unit of Model B is \$2, how many mobiles of each model are needed to produce per week to make the maximum profit?

(b) Solve the following linear programming problem by using Simplex method

$$\begin{aligned} \min \quad & z = 4x_1 - x_2 \\ \text{s.t} \quad & 2x_1 + x_2 \leq 8 \\ & x_2 \leq 5 \\ & x_1 - x_2 \leq 4 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Model Answer

Q1: [4+4]

(a)

$$R_{sys} = [1-(1-0.95)(1-0.94)](0.98)[1-(1-0.9)^3]$$
$$\simeq 0.976 = 97.6\%$$

(b)

We use the decomposition method and we take the component 3 of reliability 0,95 as a pivot element.

$$\mathrm{R}^{+} = [1-(1-0.95)^2][1-(1-0.9)^2]$$
$$= 0.987525$$
$$\mathrm{R}^{-} = 1-(1-0.95\times 0.9)(1-0.95\times 0.9)$$
$$= 0.978975$$
$$\therefore\ \mathrm{R}_{\mathrm{sys}} = \mathrm{R}_3\mathrm{R}^{+} + (1-\mathrm{R}_3)\mathrm{R}^{-}$$
$$= 0.95\times 0.987525 + 0.05\times 0.978975$$
$$\simeq 0.987 = 98.7\%$$

Q2: [4+4]

(a)

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$
$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t),\ n = 1,2,3,\ \ldots \qquad (2)$$

Let $X(t)$ represents the size of the population, and the initial condition is

$$X(0) = 0 \quad \Rightarrow\ p_0(0) = 1$$

$$\Rightarrow\ p_n(0) = \begin{cases} 1 & ,\ n = 0 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$$(1) \Rightarrow \frac{dp_0(t)}{dt} = -\lambda p_0(t)$$

$$\therefore \quad \frac{dp_0(t)}{p_0(t)} = -\lambda dt$$

$$\int_0^t \frac{dp_0(u)}{p_0(u)} = -\lambda \int_0^t du$$

$$\left[\ln p_0(u)\right]_0^t = -\lambda t$$

$$\ln p_0(t) - \ln p_0(0) = -\lambda t$$

$\ln p_0(t) - \ln 1 = -\lambda t$, where $p_0(0) = 1$

$$\therefore \quad \ln p_0(t) = -\lambda t \Rightarrow p_0(t) = e^{-\lambda t} \qquad (3)$$

$$(2) \Rightarrow \frac{dp_n(t)}{dt} = p_{n-1}(t) - \lambda p_n(t), \quad n = 1, 2, 3, \ \ldots$$

$$\therefore \quad \frac{dp_n(t)}{dt} + \lambda p_n(t) = \lambda p_{n-1}(t)$$

Multiply both sides by $e^{\lambda t}$

$$e^{\lambda t}\left[\frac{dp_n(t)}{dt} + \lambda p_n(t)\right] = \lambda p_{n-1}(t) e^{\lambda t}$$

$$\therefore \quad \frac{d}{dt}\left[e^{\lambda t} p_n(t)\right] = \lambda p_{n-1}(t) e^{\lambda t}$$

$\therefore$ By separation of variables and Integration from 0 to t, we get

$$\int_0^t d\left[e^{\lambda x} p_n(x)\right] = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$\left[e^{\lambda x} p_n(x)\right]_0^t = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx$$

$$e^{\lambda t} p_n(t) - p_n(0) = \lambda \int_0^t p_{n-1}(x) e^{\lambda x} dx, \quad n = 1, 2, 3, \ \ldots$$

$$p_n(t) = \lambda e^{-\lambda t} \int_0^t p_{n-1}(x) e^{\lambda x} dx, \quad n = 1, 2, 3, \ \ldots \qquad (4)$$

which is a recurrence relation

$at\ n=1$

$$(4) \Rightarrow p_1(t) = \lambda e^{-\lambda t}\int_0^t p_0(x)e^{\lambda x}dx$$

$\because p_0(x) = e^{-\lambda x}$ from eq. (3)

$$\therefore p_1(t) = \lambda e^{-\lambda t}\int_0^t e^{-\lambda x}e^{\lambda x}dx$$

$$=\lambda e^{-\lambda t}\int_0^t dx$$

$$\therefore p_1(t) = \lambda t e^{-\lambda t} \qquad (5)$$

$at\ n=2$

$$(4) \Rightarrow p_2(t) = \lambda e^{-\lambda t}\int_0^t p_1(x)e^{\lambda x}dx$$

$\because p_1(x) = \lambda x e^{-\lambda x}$ from eq. (5)

$$\therefore p_2(t) = \lambda e^{-\lambda t}\int_0^t \lambda x e^{-\lambda x}e^{\lambda x}dx$$

$$=\lambda^2 e^{-\lambda t}\int_0^t x dx$$

$$\therefore p_2(t) = \lambda^2 e^{-\lambda t}\left[\frac{x^2}{2}\right]_0^t$$

$$\therefore p_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!} \qquad (6)$$

From Eqs (3), (5) and (6), we can deduce that $p_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!},\ n = 0,1,2,\ldots$

(b)

(i) For Poisson Process $\{X(t);\ t \geq 0\}$, where $X(t)$ is the random variable that represents the number of messages that arrive at the telegraph office at any time t.

$$\Pr\{X(s+t) - X(s) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!},\ k = 0,1,2,\ldots$$

$$\therefore \Pr\{X(12) - X(8) = 0\} = \frac{(3\times 4)^0 e^{-3(4)}}{0!} = e^{-12}$$

$$\approx 6.1442\times 10^{-6}$$

where $\lambda = 3,\ t = 12 - 8 = 4$ and $k = 0$

(ii) Consider T is the random variable that represents the time at which the first afternoon message arrives. Afternoon is the period between 12:00 P.M. and 12:00 A.M. i.e. $t \in (0,12)$

So, we can write

$$\Pr(T > t) = \Pr\{\text{The first afternoon message arrives after t units of time}\}$$
$$= \Pr\left\{X(t) - X(0) = 0\right\}$$
$$= \frac{\left[3(t)\right]^0 e^{-3t}}{0!}$$
$$= e^{-3t},$$

which is the survival/reliability function.

Also,

$$\Pr(T \le t) = 1 - \Pr(T > t)$$

$$\therefore \ \Pr(T \le t) = 1 - e^{-3t},$$

which is the cumulative distribution function.

$\therefore \ T \sim \ \exp(3)$

i.e. $T \sim$ exponential distribution with parameter equals 3.

Q3: [4+4]

(a)

$\because \Lambda \sim \text{lognormal}(\upsilon, \tau^2)$ where $p(\frac{\lambda_m}{k} < \Lambda < k\lambda_m) = 1 - 2\alpha = 0.90$

The median λ_m and the error factor k are defined respectively as follows.

$\lambda_m = e^{\upsilon}$ and $k = e^{u_\alpha \tau}$, u_α is the upper $\alpha\%$ percentile of standard normal distribution (i.e. $\Phi(u_\alpha) = 1 - \alpha$).

at $\alpha = 0.05$, $\Phi(u_\alpha) = 0.95 \Rightarrow u_\alpha = 1.645$ (SND).

$k = e^{u_\alpha \tau} \Rightarrow \tau = \frac{\ln k}{u_\alpha} = \frac{\ln 3}{1.645} \simeq 0.668$

$\lambda_m = e^{\upsilon} \Rightarrow \upsilon = \ln \lambda_m = \ln(6 \times 10^{-5}) = -9.721$

$\therefore \Lambda \sim \text{lognormal}(-9.721, 0.668^2)$

So, the failure rate λ^* is $\lambda^* = e^{\upsilon + \tau^2/2} \simeq 7.5 \times 10^{-5}$ per hour.

(b)

(i) The mean time to failure of the pump is

$$\text{MTTF} = \tfrac{1}{\lambda} = \tfrac{1}{4.28 \times 10^{-4}}$$
$$\approx 2336 \text{ hours} \approx 3.2 \text{ months.}$$

The probability that the pump survives the first month ($t_1 = 730$ hours) is

$$R(t) = e^{-\lambda t} = e^{-4.28 \times 10^{-4}(730)}$$
$$\approx 0.732 = 73.2\%.$$

(ii) The probability that the pump will fail during the next month ($t_2 = 730$ hours), knowing that the pump has been functioning without failure during its first month is

$\Pr(T \le t_1 + t_2 | T > t_1) = \Pr(T \le t_2)$

$$= 1 - e^{-4.28\times10^{-4}(730)} \approx 0.268 = 26.8\%,$$

since the pump is as good as new when it is still functioning at time t_1 (memoryless property of exponential distribution).

Q4: [4+4]

(a)

The transition probability matrix can be written as

$$\mathbf{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-\alpha & \alpha \\ 1 & \alpha & 1-\alpha \end{array}$$

(i) The probability that no error occurs up to stage $n = 2$ is given as follows.

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00} P_{00}$$
$$= 1 \times (1-\alpha) \times (1-\alpha)$$
$$= (1-\alpha)^2$$

where $p_0 = pr(X_0 = 0) = 1$

(ii) The probability that a correct signal is received at stage 2 is given as follows.

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} + \Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$$
$$= p_0 P_{00} P_{00} + p_0 P_{01} P_{10}$$
$$= (1-\alpha)^2 + \alpha^2$$
$$= 1 - 2\alpha + 2\alpha^2$$

Another solution

(ii) The probability that a correct signal is received at stage 2 is also given as follows.

$$\Pr\{X_2 = 0 | X_0 = 0\} = p_{00}^2$$
$$= [1-\alpha \quad \alpha]\begin{bmatrix} 1-\alpha \\ \alpha \end{bmatrix}$$
$$= (1-\alpha)^2 + \alpha^2$$
$$= 1 - 2\alpha + 2\alpha^2$$

(b)

For transition probability matrix of a Markov chain $\{Y_n,\ n \ge 0\}$.

The elements of first row are given by

$$P_{0,0} = \Pr\left\{Y_{n+1} = 0 \middle| Y_n = 0\right\} = p_{00}p_{00} = 0.03(0.03) = 0.0009$$
$$P_{0,1} = \Pr\left\{Y_{n+1} = 1 \middle| Y_n = 0\right\} = p_{01}p_{00} + p_{00}p_{01} = 0.97(0.03) + 0.03(0.97) = 0.0582$$
$$P_{0,2} = \Pr\left\{Y_{n+1} = 2 \middle| Y_n = 0\right\} = p_{01}p_{01} = (0.97)^2 = 0.9409$$

The elements of second row are given by

$$P_{1,0} = \Pr\left\{Y_{n+1} = 0 \middle| Y_n = 1\right\} = p_{10}p_{00} = 0.02(0.03) = 0.0006$$
$$P_{1,1} = \Pr\left\{Y_{n+1} = 1 \middle| Y_n = 1\right\} = p_{10}p_{01} + p_{11}p_{00} = 0.02(0.97) + 0.98(0.03) = 0.0488$$
$$P_{1,2} = \Pr\left\{Y_{n+1} = 2 \middle| Y_n = 1\right\} = p_{01}p_{11} = 0.97(0.98) = 0.9506$$

The elements of third row are given by

$$P_{2,0} = \Pr\left\{Y_{n+1} = 0 \middle| Y_n = 2\right\} = p_{10}p_{10} = (0.02)^2 = 0.0004$$
$$P_{2,1} = \Pr\left\{Y_{n+1} = 1 \middle| Y_n = 2\right\} = p_{11}p_{10} + p_{10}p_{11} = 0.98(0.02) + 0.02(0.98) = 0.0392$$
$$P_{2,2} = \Pr\left\{Y_{n+1} = 2 \middle| Y_n = 2\right\} = p_{11}p_{11} = (0.98)^2 = 0.9604$$

So, the transition probability matrix for the Markov chain $\{Y_n,\ n \geq 0\}$ with state space $\{0,1,2\}$ is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.0009 & 0.0582 & 0.9409 \\ 0.0006 & 0.0488 & 0.9506 \\ 0.0004 & 0.0392 & 0.9604 \end{array} \right\| \end{array}$$

Q5: [4+4]

(a)

The LP pb. is of the form

$\max \quad 3x_1 + 2x_2$

s.t

$$5x_1 + 2x_2 \leq 900$$
$$8x_1 + 10x_2 \leq 2800$$
$$x_1 \geq 0 \quad , \quad x_2 \geq 0$$

$\Rightarrow$

The canonical form is

$$\max \quad 3x_1 + 2x_2$$

$$5x_1 + 2x_2 + x_3 = 900$$

$$8x_1 + 10x_2 + x_4 = 2800$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0$$

where x_3 and x_4 are slack variables.

Let $x_1 = x_2 = 0 \quad \Rightarrow$ NBVs = $\{x_1, x_2\}$ and BVs = $\{x_3, x_4\}$

$$\Rightarrow \quad \begin{aligned} x_3 &= 900 - 5x_1 - 2x_2 \\ x_4 &= 2800 - 8x_1 - 10x_2 \\ z &= 0 + 3x_1 + 2x_2 \end{aligned}$$

1st dictionary

Let x_1 be incoming variable (it has a +ve coefficient in the equation for z).

Ratio test

$x_3 : \dfrac{900}{5} = 180, \quad x_4 : \dfrac{2800}{8} = 350$

Min. ratio for x_3

$\therefore \; x_3 \rightarrow$ outgoing variable

$\Rightarrow \; x_1 = 180 - 2/5 \; x_2 - 1/5 \; x_3$

$x_4 = 1360 - 34/5 \; x_2 + 8/5 \; x_3$

$$\Rightarrow \quad \begin{aligned} x_1 &= 180 - 2/5 \; x_2 - 1/5 \; x_3 \\ x_4 &= 1360 - 34/5 \; x_2 + 8/5 \; x_3 \\ z &= 540 + 4/5 \; x_2 - 3/5 \; x_3 \end{aligned}$$

2nd dictionary

Let x_2 be incoming variable (it has a +ve coefficient In the equation for z).

Ratio test

$x_1 : \dfrac{180}{2/5} = 450, \quad x_4 : \dfrac{1360}{34/5} = 200$

Min. ratio for x_4

$\therefore \; x_4 \rightarrow$ outgoing variable

$\Rightarrow \; x_2 = 200 + 4/17 \; x_3 - 5/34 \; x_4$

$x_1 = 180 - \frac{2}{5}(200 + 4/17 \; x_3 - 5/34 \; x_4) - 1/5 x_3$

$$x_1 = 100 - 5/17\ x_3 + 5/85\ x_4$$
$$\Rightarrow\ x_2 = 200 + 4/17\ x_3 - 5/34\ x_4$$
$$z = 700 - 7/17\ x_3 - 2/17\ x_4$$

3rd dictionary

Here, we have –ve coefficients for all variables in the z equation, so we should stop.

$\therefore$ The optimal solution is $x_1 = 100,\ x_2 = 200$ where $\max\ z = \$700$

(b)

The LP pb. is of the form

$$\min\ \ 4x_1 - x_2$$

s.t

$$\begin{aligned} 2x_1 + x_2 &\le 8 \\ x_2 &\le 5 \\ x_1 - x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

or

$$\max\ \ -4x_1 + x_2$$

s.t

$$\begin{aligned} 2x_1 + x_2 &\le 8 \\ x_2 &\le 5 \\ x_1 - x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

The canonical form is

$$\max\ \ -4x_1 + x_2$$

s.t

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 8 \\ x_2 + x_4 &= 5 \\ x_1 - x_2 + x_5 &= 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

where x_3, x_4 and x_5 are slack variables.

Let $x_1 = x_2 = 0$

i.e. x_1 and x_2 are non-basic variables NBVs

but x_3, x_4 and x_5 are basic variables BVs

i.e. $\{x_3, x_4, x_5\}$ is the basis

So, we can write the dictionary as follows

$$\Rightarrow \quad \begin{aligned} x_3 &= 8 - 2x_1 - x_2 \\ x_4 &= 5 - x_2 \\ x_5 &= 4 - x_1 + x_2 \\ z &= -4x_1 + x_2 \end{aligned}$$

1st dictionary

We pick a variable x_2 to be **incoming variable** which having a +ve coefficient in the Eq. for z.

Ratio test

$x_3 : \frac{8}{1} = 8, \quad x_4 : \frac{5}{1} = 5 \quad x_5$: positive coefficient for x_2 (no constraint)

$\therefore\ x_4 \rightarrow$ outgoing variable

$$\Rightarrow \begin{aligned} x_2 &= 5 - x_4 \\ x_3 &= 3 - 2x_1 + x_4 \\ x_5 &= 9 - x_1 - x_4 \\ z &= 5 - 4x_1 - x_4 \end{aligned}$$

2nd dictionary

We notice that there are no +ve coefficients in the z Eq.

So, the optimal solution is $x_1 = 0,\ x_2 = 5,\ x_3 = 3,\ x_4 = 0,\ x_5 = 9$

where $z = 5$ i.e. $\max z = 5$ and consequently the min of $4x_1 - x_2$ is -5.