King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1/1447
M 507 - Advanced Operations Research
Time: 3 hours – **Marks: 40**

## Answer the following questions.

### Q1: [3+3+4]

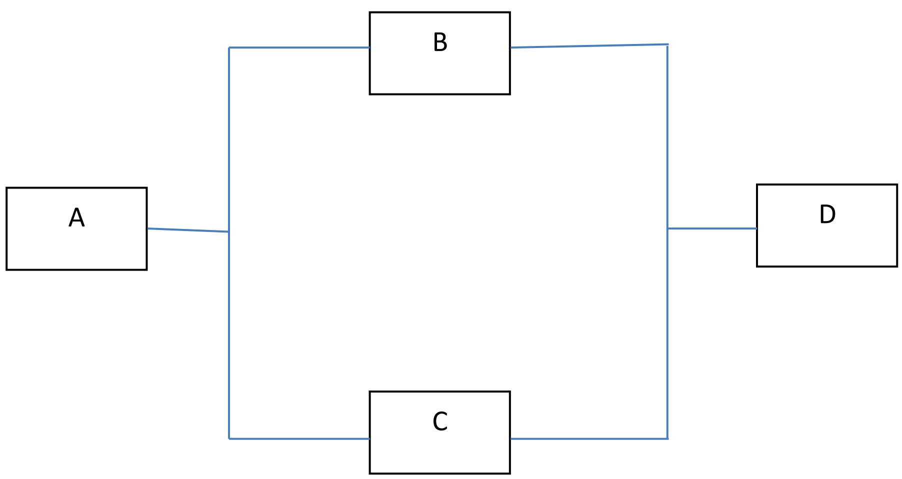
In the reliability diagram below, the reliability of each component is constant and independent. Assuming that each has the same reliability $R$, compute the system reliability as a function of $R$ using the following methods:

(a) Decomposition using $B$ as the keystone element.

(b) The reduction method.

(c) Compute the importance of each component if $R_A = 0.8$, $R_B = 0.9$,

$R_C = 0.95$ and $R_D = 0.98$




### Q2: [3+4]

(a) If a Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5$, $p_1 = 0.2$ and $p_2 = 0.3$ Find $pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$

(b) If $X(t)$ represents a size of a population where $X(0)=1$, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

Prove that:

$X(t) \sim geom\ (p)$, $p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

**Q3: [4+4]**

(a) Let $T|\Lambda$ have an exponential distribution with parameter $\lambda$. Let $\Lambda$ have a gamma distribution with parameters $k$ and $\alpha$. Determine the probability density function for the unconditional distribution of $T$, then find the survivor function $R(t)$, the failure rate function $Z(t)$, and the MTTF.

Hint: If $X \sim gamma(k,\alpha)$ then $f(x) = \dfrac{\alpha^k x^{k-1} e^{-\alpha x}}{\Gamma(k)}$ $\quad x>0,\ \alpha > 0,\ k > 0$

(b) If $X$ is the life of an item of a product and $X \sim$ Lognormal$(\mu,\ \sigma^2)$ where $\mu = 6.908$, $\sigma = 0.317$. Find the mean time to failure $\mathrm{MTTF}$, variance, failure rate at 500 hours, and also, determine the probability that the item will survive until age 500 hours.

**Q4: [6]**

Let $X_n$ denote the condition of a machine at the end of period $n$ for $n = 1, 2, \ldots$. Let $X_0$ be the condition of the machine at the start. Consider the condition of the machine at any time can be observed and classified as being in one of the following three states:

State 1: Good operating order, State 2: Deteriorated operating order and State 3: In repair.

Assume that $\{X_n\}$ is a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{ccc} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{array} \right\| \end{array}$$

and starts in state $X_0 = 1$.

(i) Find $\Pr\{X_4 = 1\}$.

(ii) Calculate the limiting distribution.

(iii) What is the long run rate of repairs per unit time?

**Q5: [4+5]**

(a) Wild West produces two types of cowboy hats. A type 1 hat requires twice as much labor time as a type 2. If the all available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The profit is $8 per Type 1 hat and $5 per Type 2 hat. Determine the number of hats of each type that would maximize profit.

(b) Solve the problem:

$$\text{Minimize } f(\mathrm{X}) = x_1^2 + x_2^2 + x_3^2$$

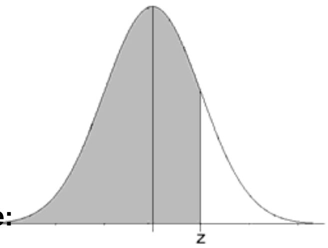
subject to

$$g_1(\mathrm{X}) = x_1 + x_2 + 3x_3 - 2 = 0$$

$$g_2(\mathrm{X}) = 5x_1 + 2x_2 + x_3 - 5 = 0$$

Consider $x_3$ as independent variable, and apply the sufficiency condition to determine the type of the resulting stationary point.

## Standard Normal Cumulative Probability Table




**Cumulative probabilities for POSITIVE z-values are shown in the following table:**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| **3.1** | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| **3.2** | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| **3.3** | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| **3.4** | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

# Model Answer

## Q1: [3+3+4]

In the reliability diagram shown in Fig. 1, the reliability of each component is constant and independent. Assuming that each has the same reliability $R$, compute the system reliability as a function of $R$ using the following methods:

a) Decomposition using B as the keystone element.

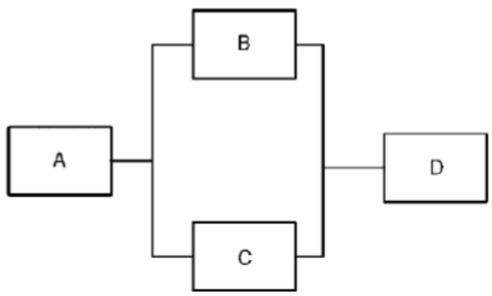



Fig. 1: Reliability diagram

Using B as the keystone element, we have two cases i.e., the case when B functions and the case when it does not.

For the case when B functions, the system reduced to Fig 2.

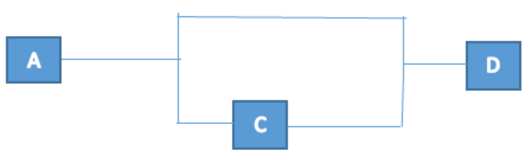



Fig. 2: The case when B functions

Thus the reliability of the system depends only on the reliability of component A and D. Note that $R_A = R_B = R_C = R_D = R$

Therefore,

$$R^{+} = R_A R_D = R^2$$

For the case when B fails, the system block is as shown in Fig. 3, which is a series system.

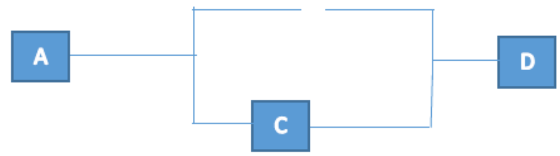



Fig. 3: The case when B fails to work

Thus the reliability of the system depends on A, C, and D, therefore we have:

$$R^{-} = R_A R_C R_D = R^3$$

Thus the reliability of the system using the two decompositions is given as:

$$R_{system} = R_B R^{+} + (1 - R_B) R^{-}$$
$$R_{system} = R(R^2) + (1 - R) R^3$$
$$R_{system} = 2R^3 - R^4$$

b) Using the reduction method

With this method, it can be seen that components B and C are in parallel and jointly in series with A and D. therefore the reduced system is given in Fig. 4.


Fig. 4: Reduced system

For parallel components B and C, we have

$$R_{B\|C} = 1 - \prod_{i=1}^{2} (1 - R_i)$$
$$R_{B\|C} = R_B + R_C - R_B R_C$$
$$R_{B\|C} = 2R - R^2$$

The reliability of the system is thus given as:

$$R_{system} = R_A R_{B\|C} R_D$$
$$R_{system} = R(2R - R^2) R$$
$$R_{system} = 2R^3 - R^4$$

c)

Recall that the reliability of the system is given as:

$$R_{system} = R_A R_D (R_B + R_C - R_B R_C)$$

The importance of each component is computed by taking the partial derivative with respect to each of the component.

Thus the importance of component A is given as:

$$\frac{\delta R_{system}}{\delta R_A} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_A}$$

$$I_A = R_D (R_B + R_C - R_B R_C)$$

$$\Rightarrow \ I_A = 0.98(0.9 + 0.95 - 0.9 \times 0.95)$$

$$= 0.9751$$

The importance of component B is given as:

$$\frac{\delta R_{system}}{\delta R_B} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_B}$$

$$I_B = R_A R_D - R_A R_D R_C$$

$$\Rightarrow \ I_B = 0.8(0.98) - 0.8(0.98)(0.95)$$

$$= 0.0392$$

The importance of component C is given as:

$$\frac{\delta R_{system}}{\delta R_C} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_C}$$

$$I_C = R_A R_D - R_A R_B R_D$$

$$\Rightarrow \ I_c = 0.8(0.98) - 0.8(0.9)(0.98)$$

$$= \ 0.0784$$

The importance of component D is given as:

$$\frac{\delta R_{system}}{\delta R_D} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_D}$$

$$I_D = R_A (R_B + R_C - R_B R_C)$$

$$\Rightarrow \ I_D = 0.8(0.9 + 0.95 - 0.9 \times 0.95)$$

$$= 0.796$$

## Q2: [3+4]

(a)

$$\text{pr}\{X_1=1, X_2=1, X_3=0\} = p_1 P_{11} P_{10}\ , \quad p_1 = \text{pr}\{X_1=1\}$$

$$\text{pr}\{X_1=1\} = \Pr(X_1=1|X_0=0)\Pr(X_0=0) + \Pr(X_1=1|X_0=1)\Pr(X_0=1) + \Pr(X_1=1|X_0=2)\Pr(X_0=2)$$

$$= P_{01}p_0 + P_{11}p_1 + P_{21}p_2$$

$$= 0.3(0.5) + 0.2(0.2) + 0.3(0.3) = 0.28$$

$$\therefore\ pr\{X_1=1, X_2=1, X_3=0\} = \ 0.28(0.2)(0.4) = 0.0224$$

(b)

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t),\ n=1,2,3,\ \ldots \qquad (2)$$

The initial condition is $X(0)=1 \quad \Rightarrow\ p_1(0)=1$

$$\Rightarrow\ p_n(0) = \begin{cases} 1 & , n=1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\lambda_0 = 0 \qquad (1) \Rightarrow \frac{dp_0(t)}{dt} = 0$$

$$\Rightarrow\ p_0(t) = 0 \qquad (3)$$

$$(2) \Rightarrow\ \frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t)$$

$$\Rightarrow\ \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t),\ n=1,2,\ \ldots$$

$$\because\ \lambda_n = n\lambda,\ \ \lambda_{n-1} = (n-1)\lambda$$

$$\therefore\ \frac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t),\ \text{n}=1,2,\ \ldots$$

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt}+n\lambda p_n(t)\right]=(n-1)\lambda p_{n-1}(t)e^{n\lambda t}$$

$$\therefore\ \frac{d}{dt}\left[p_n(t)e^{n\lambda t}\right]=(n-1)\lambda p_{n-1}(t)e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x)e^{n\lambda x}\right]=(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx$$

$$\therefore\ \left[p_n(x)e^{n\lambda x}\right]_0^t=(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx$$

$$\Rightarrow\ p_n(t)=e^{-n\lambda t}\left[p_n(0)+(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx\right],\ n=1,2,\ \dots \quad (4)$$

which is a recurrence relation.

at $n=1$

$$p_1(t)=e^{-\lambda t}\left[p_1(0)+0\right]=e^{-\lambda t} \qquad (5)$$

at $n=2$

$$p_2(t)=e^{-2\lambda t}\left[p_2(0)+\lambda\int_0^t p_1(x)e^{2\lambda x}dx\right]$$

$(5)\Rightarrow\ p_1(x)=e^{-\lambda x}$

$$\therefore\ p_2(t)=e^{-2\lambda t}\left[\lambda\int_0^t e^{-\lambda x}e^{2\lambda x}dx\right]$$

$$\therefore\ p_2(t)=\lambda e^{-2\lambda t}\int_0^t e^{\lambda x}dx$$

$$=e^{-\lambda t}(1-e^{-\lambda t})^1 \qquad (6)$$

Similarly as (5) and (6), we deduce that

$$p_n(t)=e^{-\lambda t}(1-e^{-\lambda t})^{n-1}$$

$$=p(1-p)^{n-1},\quad p=e^{-\lambda t},\ n=1,2,\ \dots$$

$$\therefore X(t)\sim geom(p),\ p=e^{-\lambda t}$$

$$Mean\left[X(t)\right]=1/p=e^{\lambda t},$$

$$Variance\left[X(t)\right]=\frac{1-p}{p^2}=\frac{1-e^{-\lambda t}}{e^{-2\lambda t}}$$

## Q3: [4+4]

(a) $f(t|\lambda)=\lambda e^{-\lambda t} \quad t>0 \quad (1)$

$\because \Lambda \sim gamma(k,\alpha)$

$$\therefore \pi(\lambda) = \frac{\alpha^k \lambda^{k-1} e^{-\alpha\lambda}}{\Gamma(k)} \quad \lambda>0,\ \alpha>0,\ k>0 \quad (2)$$

The unconditional or mixture or marginal density function of $T$ is given by

$$f(t) = \int_0^\infty f(t|\lambda)\pi(\lambda)d\lambda \quad (3)$$

$$f(t) = \frac{\alpha^k}{\Gamma(k)}\int_0^\infty \lambda^k e^{-(\alpha+t)\lambda}d\lambda$$

$$= \frac{\alpha^k}{\Gamma(k)(\alpha+t)^{k+1}}\int_0^\infty u^k e^{-u}du, \quad u=(\alpha+t)\lambda$$

$$\therefore\ f(t) = \frac{\Gamma(k+1)\alpha^k}{\Gamma(k)(\alpha+t)^{k+1}} = \frac{k\alpha^k}{(\alpha+t)^{k+1}} \quad (4)$$

The survivor function is

$$R(t) = \Pr(T>t) = \int_t^\infty f(u)du$$

$$= \int_t^\infty \frac{k\alpha^k}{(\alpha+u)^{k+1}}du = k\alpha^k\int_t^\infty (\alpha+u)^{-k-1}du$$

$$\therefore R(t) = -\frac{\alpha^k}{(\alpha+u)^k}\Bigg|_t^\infty = \left(\frac{\alpha}{\alpha+t}\right)^k = \left(1+\frac{t}{\alpha}\right)^{-k} \quad (5)$$

The mean time to failure is

$$MTTF = \int_0^\infty R(t)dt$$

$$= \int_0^\infty \left(1+\frac{t}{\alpha}\right)^{-k} dt$$

$$= -\alpha\frac{(1+t/\alpha)^{1-k}}{k-1}\Bigg|_0^\infty$$

$$\therefore MTTF = \frac{\alpha}{k-1} \quad k>1 \quad (6)$$

Note that $MTTF$ does not exist for $0 \le k \le 1$.

The failure rate function is $Z(t) = \dfrac{f(t)}{R(t)} = \dfrac{k}{\alpha+t}$ which is a decreasing function with $t$.

(b) For Lognormal distribution

$$MTTF = e^{\mu+\sigma^2/2}$$
$$= e^{[6.908+0.5(0.317)^2]}$$
$$= 1051.785526$$

$$Var(X) = \exp(2\mu)[\exp(2\sigma^2) - \exp(\sigma^2)]$$

or

$$Var(X) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$$
$$= \exp(2 \times 6.908 + 0.317^2)[\exp(0.317^2) - 1]$$
$$= 116943.6187$$

The failure rate at 500 hours is $Z(500)$,

$$Z(x) = \frac{f(x)}{R(x)}$$
$$= \frac{\frac{1}{\sigma x}\varphi\left[\frac{\ln x - \mu}{\sigma}\right]}{1 - \Phi\left[\frac{\ln x - \mu}{\sigma}\right]}$$
$$= \frac{\frac{1}{\sigma x}\frac{1}{\sqrt{2\pi}\sigma}e^{-(\ln x - \mu)^2/2\sigma^2}}{1 - \Phi\left[\frac{\ln x - \mu}{\sigma}\right]}$$
$$Z(500) = \frac{7.258965119 \times 10^{-4}}{\Phi(2.19)}$$
$$= \frac{7.258965119 \times 10^{-4}}{0.9857}$$
$$= 7.3643 \times 10^{-4} \text{ hours}$$

The reliability at 500 hours is $R(500)$,

$$R(x) = 1 - \Phi\left[\frac{\ln x - \mu}{\sigma}\right]$$
$$= 1 - \Phi\left[\frac{\ln 500 - 6.908}{0.317}\right]$$
$$= 1 - \Phi[-2.19]$$
$$= \Phi(2.19)$$
$$\therefore R(500) = 0.9857$$

## Q4: [6]

(i)

$$\mathbf{P}= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left\| \begin{array}{ccc} 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{array} \right\| \end{array}$$

$$\because \ \Pr\{X_4 = 1\} = \Pr\{X_4 = 1 | X_0 = 1\}\Pr\{X_0 = 1\}$$
$$= P_{11}^4, \ \Pr\{X_0 = 1\} = 1$$

$$\mathrm{P}^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left\| \begin{array}{ccc} 0.81 & 0.18 & 0.01 \\ 0.1 & 0.81 & 0.09 \\ 0.9 & 0.1 & 0 \end{array} \right\| \end{array}$$

and $$\mathrm{P}^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left\| \begin{array}{ccc} 0.6831 & 0.2926 & 0.0243 \\ 0.2430 & 0.6831 & 0.0739 \\ 0.7390 & 0.2430 & 0.0180 \end{array} \right\| \end{array}$$

$$\therefore \Pr\{X_4 = 1\} = \Pr\{X_4 = 1 | X_0 = 1\}\Pr\{X_0 = 1\}$$
$$= P_{11}^4 p_1 = P_{11}^4 = 0.6831$$

(ii) To get the limiting distribution $\pi = (\pi_1, \pi_2, \pi_3) = (\pi_G, \pi_D, \pi_R)$

Solving the following equations

$$\pi_1 = 0.9\pi_1 + \pi_3 \qquad (1)$$
$$\pi_2 = 0.1\pi_1 + 0.9\pi_2 \qquad (2)$$
$$\pi_3 = 0.1\pi_2 \qquad (3)$$
$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

$(1) \Rightarrow \pi_3 = 0.1\pi_1$
$(2) \Rightarrow \pi_2 = \pi_1$
also, (3) $\pi_3 = 0.1\pi_2$
$(4) \Rightarrow \pi_1 + \pi_1 + 0.1\pi_1 = 1$
$\therefore \ \pi_1 = \frac{10}{21}$
$\Rightarrow \therefore \ \pi_2 = \frac{10}{21}$ and $\pi_3$

$\therefore\ \pi = (\frac{10}{21}, \frac{10}{21}, \frac{1}{21})$

(iii) $\pi_R = \pi_3 = \frac{1}{21} = 0.0476$

## Q5: [4+5]

(a) Let $x_1$ is the daily # of type 1 hat and $x_2$ is the daily # of type 2 hat

The LP problem will be as follows:

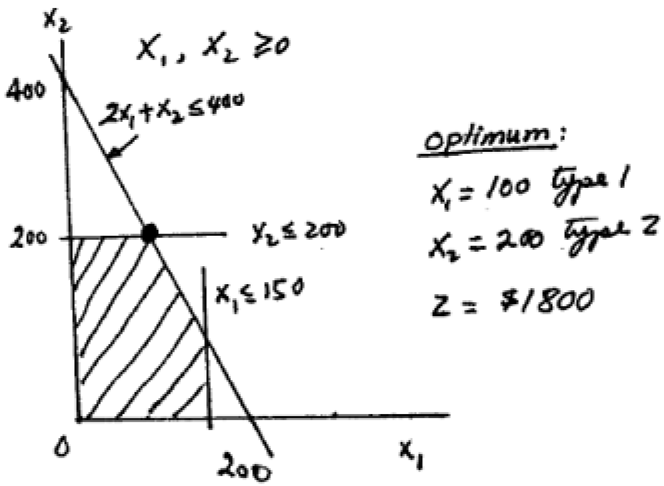
$$\max \quad z = 8x_1 + 5x_2$$

$$\text{s.t} \quad 2x_1 + x_2 \le 400$$

$$x_1 \le 150$$

$$x_2 \le 200$$

$$x_1 \ge 0\ ,\quad x_2 \ge 0$$




$\therefore$ The optimal solution is $x_1 = 100,\ x_2 = 200$ where $\max\ z = \$1800$

(b) We determined the constrained extreme points as follows.

Let $\mathrm{Y} = (x_1 \quad x_2)$ and $\mathrm{Z} = x_3$ thus,

$$\nabla_{\mathrm{Y}} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix}$$

$$\nabla_{\mathrm{Z}} f = \frac{\partial f}{\partial x_3} = 2x_3$$

The Jacobian matrix is $\mathrm{J} = \nabla_{\mathrm{Y}} g = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix}$

$J^{-1}=\begin{pmatrix}-\frac{2}{3} & \frac{1}{3}\\ \frac{5}{3} & -\frac{1}{3}\end{pmatrix}$ and the control matrix is $C=\nabla_Z g=\begin{pmatrix}\frac{\partial g_1}{\partial x_3}\\ \frac{\partial g_2}{\partial x_3}\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$

The constrained gradient vector of $f$ is $\nabla_c f=\frac{\partial_c f}{\partial_c x_3}=\nabla_Z f-\nabla_Y f\ J^{-1}C$

$$\therefore \nabla_c f=2x_3-\begin{pmatrix}2x_1 & 2x_2\end{pmatrix}\begin{pmatrix}-\frac{2}{3} & \frac{1}{3}\\ \frac{5}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}$$
$$=\tfrac{10}{3}x_1-\tfrac{28}{3}x_2+2x_3$$

To obtain the stationary points, solve the following three equations

$\nabla_c f=0,\ g_1(X)=0$ and $g_2(X)=0$, which can be written as

$$10x_1-28x_2+6x_3=0$$
$$x_1+x_2+3x_3=2$$
$$5x_1+2x_2+x_3=5$$

Solving these equations by using Cramer's rule

$$x_1=\frac{\det(A_1)}{\det(A)},x_2=\frac{\det(A_2)}{\det(A)}\text{ and }x_3=\frac{\det(A_3)}{\det(A)},\text{ where}$$
$$\det(A)=-460,\ \det(A_1)=-370,\ \det(A_2)=-160\text{ and }\det(A_3)=-130$$

Therefore, the solution is $X_0=\begin{pmatrix}\frac{37}{46} & \frac{8}{23} & \frac{13}{46}\end{pmatrix}$ which is the stationary point.

To identify this stationary point whether min. or. Max. use the sufficiency condition as follows (given that $x_3$ as independent variable)

$$\nabla_c f=\tfrac{\partial f}{\partial x_3}=\tfrac{10}{3}x_1-\tfrac{28}{3}x_2+2x_3$$
$$\frac{\partial^2 f}{\partial x_3^2}=\tfrac{10}{3}\left(\tfrac{dx_1}{dx_3}\right)-\tfrac{28}{3}\left(\tfrac{dx_2}{dx_3}\right)+2$$
$$=\begin{pmatrix}\frac{10}{3} & -\frac{28}{3}\end{pmatrix}\begin{pmatrix}\frac{dx_1}{dx_3}\\ \frac{dx_2}{dx_3}\end{pmatrix}+2$$

$$\because \partial Y=-J^{-1}C\partial Z$$

$$i.e.\ \begin{pmatrix}\partial x_1\\ \partial x_2\end{pmatrix}=-J^{-1}C\partial x_3$$

$$\therefore \begin{pmatrix}\frac{dx_1}{dx_3}\\ \frac{dx_2}{dx_3}\end{pmatrix}=-J^{-1}C=-\begin{pmatrix}-\frac{2}{3} & \frac{1}{3}\\ \frac{5}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}\frac{5}{3}\\ \frac{-14}{3}\end{pmatrix}$$

$$\therefore \frac{\partial^2 f}{\partial x_3^2}=\begin{pmatrix}\frac{10}{3} & -\frac{28}{3}\end{pmatrix}\begin{pmatrix}\frac{5}{3}\\ \frac{-14}{3}\end{pmatrix}+2=\tfrac{460}{9}>0$$

Hence, $X_0=\begin{pmatrix}\frac{37}{46} & \frac{8}{23} & \frac{13}{46}\end{pmatrix}$ is the minimum point.

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