Solution to Final Exam Math 487/471

(1) Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+4)^2}$$

Consider the complex function

$$g(z) = \frac{ze^{iz}}{(z^2+4)^2}$$

and the contour consisting of the real axis from $-R$ to R and the upper semicircle of radius R. The only pole inside the upper half-plane is at $z = 2i$,

which is a pole of order 2.

Compute the residue:

$$\operatorname{Res}(g, 2i) = \lim_{z \to 2i} \frac{d}{dz} \left((z-2i)^2 g(z) \right) = \frac{e^{-2}}{8}.$$

By the residue theorem,

$$\int_{-R}^{R} g(x) \, dx + \int_{\text{semicircle}} g(z) \, dz = 2\pi i \cdot \frac{e^{-2}}{8} = \frac{\pi i e^{-2}}{4}.$$

As $R \to \infty$, the integral over the semicircle vanishes. Hence,

$$\left| \int_\gamma g(z) dz \right| \leq \frac{\pi R^2}{(R^2-4)^2} \xrightarrow{R \to \infty} 0$$

Hence,

$$\int_{-\infty}^{\infty} g(x) \, dx = \frac{\pi i e^{-2}}{4}.$$

The imaginary part gives the desired integral, so

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+4)^2} = \operatorname{Im} \left(\frac{\pi i e^{-2}}{4} \right) = \frac{\pi}{4e^2}.$$

(2) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta\cos\theta}$$

Simplify $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$. Then

$$I = \int_0^{2\pi} \frac{d\theta}{2+\frac{1}{2}\sin 2\theta} = \int_0^{2\pi} \frac{2\,d\theta}{4+\sin 2\theta}.$$

Let $\varphi = 2\theta$, $d\theta = d\varphi/2$. As θ runs from 0 to 2π, φ runs from 0 to 4π. Thus,

$$I = \int_0^{4\pi} \frac{d\varphi}{4+\sin\varphi} = 2\int_0^{2\pi} \frac{d\varphi}{4+\sin\varphi} = 2J,$$

where $J = \int_0^{2\pi} \frac{d\varphi}{4+\sin\varphi}$. Use the substitution $z = e^{i\varphi}$, so that

$$\sin\varphi = \frac{z-z^{-1}}{2i}, \quad d\varphi = \frac{dz}{iz}.$$

Then

$$J = \oint_{|z|=1} \frac{2\,dz}{z^2+8iz-1}.$$

The poles are at $z = i(-4\pm\sqrt{15})$. Only $z_1 = i(\sqrt{15}-4)$ lies inside $|z|=1$. The residue at z_1 is

$$\operatorname{Res}\left(\frac{2}{z^2+8iz-1}, z_1\right) = \frac{2}{2z_1+8i} = \frac{1}{z_1+4i} = \frac{1}{i\sqrt{15}} = -\frac{i}{\sqrt{15}}.$$

Hence,

$$J = 2\pi i\left(-\frac{i}{\sqrt{15}}\right) = \frac{2\pi}{\sqrt{15}}, \quad \text{so} \quad I = 2J = \frac{4\pi}{\sqrt{15}}.$$

(3) Find Laurent expansion of $f(z) = \frac{1}{z^3-25z}$ in the infinite annular $\{z\colon |z| > 5\}$. Use expansion to find $\int_\gamma \frac{z^3 dz}{z^3-25z}$, where γ is the circle $|z| = 6$, in the positive direction.

Factor:

$$f(z) = \frac{1}{z(z-5)(z+5)} = \frac{1}{z^3} \cdot \frac{1}{1 - 25/z^2}, \quad |z| > 5.$$

Expand geometrically:

$$f(z) = \frac{1}{z^3} \sum_{n=0}^{\infty} \left(\frac{25}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{25^n}{z^{2n+3}}.$$

Then

$$\frac{z^3}{z^3 - 25z} = z^3 f(z) = \sum_{n=0}^{\infty} \frac{25^n}{z^{2n}} = 1 + \frac{25}{z^2} + \frac{625}{z^4} + \cdots .$$

Alternatively,

$$\frac{1}{z(z-5)(z+5)} = \frac{A}{z} + \frac{B}{z-5} + \frac{C}{z+5}$$

$$1 = A(z-5)(z+5) + Bz(z+5) + Cz(z-5)$$

Let $z = 0$:

$$1 = A(-5)(5) + 0 + 0$$

$$1 = A(-25) \quad \Rightarrow \quad A = -\frac{1}{25}$$

Let $z = 5$:

$$1 = 0 + B \cdot 5 \cdot (10) + 0$$

$$1 = 50B \quad \Rightarrow \quad B = \frac{1}{50}$$

Let $z = -5$:

$$1 = 0 + 0 + C(-5)(-10)$$

$$1 = C(50) \quad \Rightarrow \quad C = \frac{1}{50}$$

$$f(z) = -\frac{1}{25} \cdot \frac{1}{z} + \frac{1}{50} \cdot \frac{1}{z-5} + \frac{1}{50} \cdot \frac{1}{z+5}$$

For $|z| > 5$, $|5/z| < 1$ and $|-5/z| < 1$:

$$\frac{1}{z-5} = \frac{1}{z} \cdot \frac{1}{1-\frac{5}{z}}$$

and

$$\frac{1}{z+5} = \frac{1}{z} \cdot \frac{1}{1+\frac{5}{z}}.$$

Expand in powers of $1/z$

$$\frac{1}{1-\frac{5}{z}} = \sum_{n=0}^{\infty} \left(\frac{5}{z}\right)^n, \quad |z| > 5$$

$$\frac{1}{1+\frac{5}{z}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{5}{z}\right)^n, \quad |z| > 5$$

So:

$$\frac{1}{z-5} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{5^n}{z^n} = \sum_{n=0}^{\infty} \frac{5^n}{z^{n+1}}$$

$$\frac{1}{z+5} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{z^n} = \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{z^{n+1}}$$

So,

$$\frac{1}{z-5} + \frac{1}{z+5} = \sum_{n=0}^{\infty} \frac{5^n + (-1)^n 5^n}{z^{n+1}}$$

But

$$5^n \left[1 + (-1)^n\right] = \begin{cases} 2 \cdot 5^n & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

So:

$$\frac{1}{z-5} + \frac{1}{z+5} = \sum_{k=0}^{\infty} \frac{2 \cdot 25^k}{z^{2k+1}}$$

Hence

$$\frac{1}{50(z-5)} + \frac{1}{50(z+5)} = \frac{1}{50}\sum_{k=0}^{\infty} \frac{2 \cdot 25^k}{z^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{25^k}{25 \cdot z^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{25^{k-1}}{z^{2k+1}}$$

The first term from partial fractions was $-\frac{1}{25z}$.
Note that in the sum above for $k = 0$, the term is $+\frac{1}{25z}$.

Hence,

$$f(z) = \sum_{k=1}^{\infty} \frac{25^{k-1}}{z^{2k+1}}$$

Or,

$$f(z) = \sum_{n=0}^{\infty} \frac{25^n}{z^{2n+3}};$$

Then

$$\frac{z^3}{z^3 - 25z} = z^3 f(z) = \sum_{n=0}^{\infty} \frac{25^n}{z^{2n}} = 1 + \frac{25}{z^2} + \frac{625}{z^4} + \cdots.$$

$$\int_\gamma \frac{z^3}{z^3 - 25z}\, dz = \int_\gamma \left(1 + \frac{25}{z^2} + \frac{625}{z^4} + \cdots\right) dz = 0,$$

since all integrands are of the form z^{-k} with $k \neq 1$ (and the integral of such terms over a closed contour is zero).

Or, by residues, the integral is zero.

3)

Given $f(z) = u(x,y) + iv(x,y)$ analytic on a domain D, we show that the mixed partial derivative $u_{xxyx}(x,y)$ exists and is continuous on D.

Since f is analytic on D, it has derivatives of all orders (see book).

Consequently, the real and imaginary parts u and v are infinitely differentiable

As u is infinitely differentiable, all partial derivatives of u of any order exist and are continuous. In particular, the fourth-order mixed partial derivative u_{xxyx} exists and is continuous on D.

Thus, $u_{xxyx}(x,y)$ exists and is continuous on D.

(5) Find the Cauchy–Riemann equations in polar coordinates and $f'(z)$ in polar form.

Let $z = re^{i\theta}$, and write $u = u(r, \theta)$, $v = v(r, \theta)$. The Cauchy–Riemann equations in polar coordinates are:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

The derivative is given by:

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).$$

See book.

(6) Compute

$$\int_\gamma \frac{\operatorname{Log} z \, dz}{z^2 - 2z + 2},$$

where $\operatorname{Log} z$ is the principal branch of the logarithm and γ is the positively oriented rectangle with vertices $\left(\frac{1}{2}, 2i\right)$, $(2, 2i)$, $(2, -2i)$, $\left(\frac{1}{2}, -2i\right)$.

The denominator vanishes at $z = 1 \pm i$, both simple poles inside γ. The principal logarithm is analytic inside γ. By the residue theorem:

$$I = 2\pi i\left(\operatorname{Res}(f, 1+i) + \operatorname{Res}(f, 1-i)\right),$$

where

$$\operatorname{Res}(f, 1+i) = \frac{\operatorname{Log}(1+i)}{2(1+i)-2} = \frac{\operatorname{Log}(1+i)}{2i}, \quad \operatorname{Res}(f, 1-i) = \frac{\operatorname{Log}(1-i)}{2(1-i)-2} = \frac{\operatorname{Log}(1-i)}{-2i}.$$

Using $\operatorname{Log}(1+i) = \frac{1}{2}\ln 2 + i\frac{\pi}{4}$ and $\operatorname{Log}(1-i) = \frac{1}{2}\ln 2 - i\frac{\pi}{4}$,

$$I = 2\pi i\left(\frac{1}{2i}\left(\frac{1}{2}\ln 2 + i\frac{\pi}{4}\right) + \frac{1}{-2i}\left(\frac{1}{2}\ln 2 - i\frac{\pi}{4}\right)\right) = \pi\left(i\frac{\pi}{2}\right) = i\frac{\pi^2}{2}.$$

(7) For $f(z) = \frac{\cosh(z^2)-1}{z^4}$, show that $z = 0$ is a removable singularity and find $f^{(10)}(0)$.

Expand $\cosh(z^2)$:

$$\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \quad \Rightarrow \quad \cosh(z^2) - 1 = \sum_{n=1}^{\infty} \frac{z^{4n}}{(2n)!}.$$

Thus,

$$f(z) = \frac{1}{z^4} \sum_{n=1}^{\infty} \frac{z^{4n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{z^{4n-4}}{(2n)!} = \sum_{k=0}^{\infty} \frac{z^{4k}}{(2k+2)!}.$$

This is a power series with no negative powers, so f is analytic at 0 (removable singularity). The Taylor series about 0 is

$$f(z) = \sum_{k=0}^{\infty} \frac{z^{4k}}{(2k+2)!}.$$

Since the series contains only powers $z^0, z^4, z^8, \ldots$, the coefficient of z^{10} is zero. Hence,

$$\frac{f^{(10)}(0)}{10!} = 0 \quad \Rightarrow \quad f^{(10)}(0) = 0.$$

Answer: $f^{(10)}(0) = 0$.