Math 481 Midterm 2

Time: 90 minutes Total: 25 points

Version with Full Solutions

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1. Let 5

$$f_n(x) = \frac{x}{1+n^2x}, \qquad x \in [0,1].$$

   (a) Find the pointwise limit $f(x)$.

   (b) Determine whether the convergence is uniform on $[0,1]$. Prove or disprove using the ε–N definition.

   (c) Decide whether

$$\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx.$$

Solution.

(a) Pointwise limit.

For $x = 0$,

$$f_n(0) = \frac{0}{1+n^2\cdot 0} = 0.$$

For $x > 0$, divide numerator and denominator by n^2:

$$f_n(x) = \frac{x}{1+n^2x} = \frac{x/n^2}{1/n^2+x}.$$

As $n \to \infty$,

$$\frac{x/n^2}{1/n^2+x} \to \frac{0}{x} = 0.$$

Thus

$$\boxed{f_n(x) \to f(x) = 0 \quad \text{for every } x \in [0,1].}$$

(b) Uniform convergence.

We prove that $f_n \to 0$ uniformly on $[0,1]$. Since $f_n(x) \geq 0$, we estimate

$$0 \leq f_n(x) = \frac{x}{1+n^2x}.$$

Because $1 + n^2x \geq n^2x$, for $x > 0$,

$$\frac{x}{1+n^2x} \leq \frac{x}{n^2x} = \frac{1}{n^2}.$$

At $x = 0$, the same inequality is true because $f_n(0) = 0$. Hence

$$|f_n(x) - 0| \le \frac{1}{n^2} \qquad \text{for all } x \in [0, 1].$$

Now let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\varepsilon}}.$$

Then for all $n \ge N$,

$$\frac{1}{n^2} \le \frac{1}{N^2} < \varepsilon.$$

Therefore, for all $x \in [0, 1]$,

$$|f_n(x) - 0| < \varepsilon.$$

This proves, by the ε–N definition, that

$$\boxed{f_n \to 0 \text{ uniformly on } [0, 1].}$$

(c) Convergence of the integrals.

Since $f_n \to f = 0$ uniformly on $[0, 1]$, we may pass the limit through the Riemann integral:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

Thus

$$\boxed{\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx = 0.}$$

We can also verify directly:

$$\int_0^1 \frac{x}{1 + n^2 x}\,dx.$$

Let $u = 1 + n^2 x$. Then $du = n^2\,dx$, $x = (u - 1)/n^2$, and

$$\int_0^1 \frac{x}{1 + n^2 x}\,dx = \frac{1}{n^4} \int_1^{1+n^2} \frac{u - 1}{u}\,du = \frac{1}{n^4} \int_1^{1+n^2} \left(1 - \frac{1}{u}\right) du.$$

Hence

$$\int_0^1 \frac{x}{1 + n^2 x}\,dx = \frac{1}{n^4} \left[u - \ln u\right]_1^{1+n^2} = \frac{n^2 - \ln(1 + n^2)}{n^4} \to 0.$$

Final Answer

$$f(x) = 0, \qquad f_n \to f \text{ uniformly on } [0, 1], \qquad \int_0^1 f_n\,dx \to 0 = \int_0^1 f\,dx.$$

2. Show that 3

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}\, dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^4}.$$

Justify carefully the interchange between the integral and the infinite series, citing an appropriate convergence theorem.

Solution.

For every $x \in [0, \pi]$, we have

$$\left| \frac{\sin(nx)}{n^3} \right| \le \frac{1}{n^3}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty,$$

the Weierstrass M-test implies that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$$

converges uniformly on $[0, \pi]$. Therefore we may integrate the series term-by-term:

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}\, dx = \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\sin(nx)}{n^3}\, dx.$$

Now compute each term:

$$\int_0^{\pi} \frac{\sin(nx)}{n^3}\, dx = \frac{1}{n^3} \int_0^{\pi} \sin(nx)\, dx.$$

Since

$$\int \sin(nx)\, dx = -\frac{\cos(nx)}{n},$$

we get

$$\frac{1}{n^3} \int_0^{\pi} \sin(nx)\, dx = \frac{1}{n^3} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{1}{n^4} \left(1 - \cos(n\pi) \right).$$

But $\cos(n\pi) = (-1)^n$, so

$$\int_0^{\pi} \frac{\sin(nx)}{n^3}\, dx = \frac{1 - (-1)^n}{n^4}.$$

If n is even, then $(-1)^n = 1$, and the term is 0. If n is odd, write $n = 2k - 1$. Then $(-1)^n = -1$, and

$$\frac{1 - (-1)^n}{n^4} = \frac{2}{(2k-1)^4}.$$

Therefore

$$\int_0^\pi \sum_{n=1}^\infty \frac{\sin(nx)}{n^3}\,dx = \sum_{k=1}^\infty \frac{2}{(2k-1)^4}.$$

Renaming k as n,

$$\boxed{\int_0^\pi \sum_{n=1}^\infty \frac{\sin(nx)}{n^3}\,dx = \sum_{n=1}^\infty \frac{2}{(2n-1)^4}.}$$

Justification

The interchange of integral and infinite series is justified by uniform convergence, obtained from the Weierstrass M-test.

3. Prove that [3]

$$\lim_{x\to 0} \sum_{n=1}^\infty \frac{e^{-n^2x^2}}{3^n} = \frac{1}{2}.$$

Justify the interchange between the limit and the infinite sum, stating the relevant theorem.

Solution.

For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$,

$$0 < e^{-n^2x^2} \le 1.$$

Therefore

$$\left|\frac{e^{-n^2x^2}}{3^n}\right| \le \frac{1}{3^n}.$$

Since

$$\sum_{n=1}^\infty \frac{1}{3^n} = \frac{1/3}{1-1/3} = \frac{1}{2} < \infty,$$

the Weierstrass M-test implies that

$$\sum_{n=1}^\infty \frac{e^{-n^2x^2}}{3^n}$$

converges uniformly on $\mathbb{R}$. Hence the limit may pass through the infinite sum:

$$\lim_{x\to 0} \sum_{n=1}^\infty \frac{e^{-n^2x^2}}{3^n} = \sum_{n=1}^\infty \lim_{x\to 0} \frac{e^{-n^2x^2}}{3^n}.$$

For each fixed n,

$$\lim_{x\to 0} e^{-n^2x^2} = e^0 = 1.$$

Thus

$$\sum_{n=1}^{\infty} \lim_{x\to 0} \frac{e^{-n^2x^2}}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

Therefore

$$\boxed{\lim_{x\to 0} \sum_{n=1}^{\infty} \frac{e^{-n^2x^2}}{3^n} = \frac{1}{2}.}$$

Important Note

If the series starts at $n = 0$, then the limit is not $1/2$. In that case,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{3}{2}.$$

So the statement with lower index $n = 0$ would have answer $3/2$, not $1/2$.

4. Find the radius of convergence R of each of the following power series: [7]

$$\sum_{n=0}^{\infty} a_n x^n.$$

(a)

$$a_n = \begin{cases} 2, & n = k^3 \text{ for some } k \in \mathbb{N}, \\ \dfrac{1}{n}, & \text{otherwise.} \end{cases}$$

(b) Suppose there exist constants $b, c > 0$ such that for all n,

$$nb \le |a_n| \le n^2 c.$$

Determine R and justify your answer.

Solution.

For a power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

the radius of convergence is determined by the Cauchy-Hadamard formula:

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

(a) Suppose

$$a_n = \begin{cases} 2, & n = k^3 \text{ for some } k \in \mathbb{N}, \\ 1/n, & \text{otherwise.} \end{cases}$$

We need to compute

$$\limsup_{n\to\infty} |a_n|^{1/n}.$$

There are two types of indices.

If $n = k^3$, then $a_n = 2$, so

$$|a_n|^{1/n} = 2^{1/n} \to 1.$$

If n is not a cube, then $a_n = 1/n$, so

$$|a_n|^{1/n} = \left(\frac{1}{n}\right)^{1/n} = n^{-1/n} \to 1.$$

Both subsequences have limit 1. Hence

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1.$$

Therefore

$$\frac{1}{R} = 1, \qquad \text{so} \qquad \boxed{R = 1.}$$

(b) Suppose there exist constants $b, c > 0$ such that

$$nb \le |a_n| \le n^2 c.$$

Take n-th roots:

$$(nb)^{1/n} \le |a_n|^{1/n} \le (n^2 c)^{1/n}.$$

Now

$$(nb)^{1/n} = n^{1/n} b^{1/n} \to 1 \cdot 1 = 1,$$

and

$$(n^2 c)^{1/n} = n^{2/n} c^{1/n} \to 1 \cdot 1 = 1.$$

By the squeeze theorem,

$$|a_n|^{1/n} \to 1.$$

Thus

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1.$$

Therefore

$$\frac{1}{R} = 1, \qquad \text{so} \qquad \boxed{R = 1.}$$

Final Answer

$$\boxed{R = 1 \text{ in both parts.}}$$

5. Consider the power series [7]

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

(a) Determine the radius of convergence R, and deduce that the series converges uniformly on every interval $[-c, c]$ with $0 \le c < 1$. Show that f is differentiable on $(-1, 1)$.

(b) Identify explicitly the function represented by the series on $(-1, 1)$. *Hint: first compute $f'(x)$, then use the condition $f(0) = 0$.*

(c) Apply *Abel's test for uniform convergence* to discuss the uniform convergence of the series on the closed interval $[0, 1]$.

Solution.

(a) Radius of convergence and uniform convergence on compact subintervals.

The coefficients are

$$a_n = \frac{(-1)^{n-1}}{n}.$$

Thus

$$|a_n|^{1/n} = \left(\frac{1}{n}\right)^{1/n} = n^{-1/n} \to 1.$$

By the Cauchy-Hadamard formula,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = 1.$$

Therefore

$$\boxed{R = 1.}$$

Now fix $0 \le c < 1$. For $x \in [-c, c]$,

$$\left| \frac{(-1)^{n-1}}{n} x^n \right| = \frac{|x|^n}{n} \le \frac{c^n}{n} \le c^n.$$

Since

$$\sum_{n=1}^{\infty} c^n$$

converges, the Weierstrass M-test implies that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

converges uniformly on $[-c, c]$.

To show differentiability on $(-1, 1)$, differentiate term-by-term inside the radius of convergence. The derivative series is

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}}{n} x^n \right)' = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}.$$

For $|x| < 1$, this is the geometric series

$$\sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum_{m=0}^{\infty} (-x)^m = \frac{1}{1+x}.$$

Therefore f is differentiable on $(-1, 1)$, and

$$\boxed{f'(x) = \frac{1}{1+x}, \qquad |x| < 1.}$$

(b) Identifying the function.

From part (a),

$$f'(x) = \frac{1}{1+x}.$$

Integrating gives

$$f(x) = \ln(1+x) + C.$$

Since

$$f(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} 0^n = 0,$$

we get

$$0 = \ln(1+0) + C = 0 + C,$$

so $C = 0$. Therefore

$$\boxed{f(x) = \ln(1+x), \qquad -1 < x < 1.}$$

(c) Uniform convergence on $[0, 1]$ by Abel's test.

On $[0,1]$, write the series as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

Set

$$a_n(x) = x^n, \qquad b_n = \frac{(-1)^{n-1}}{n}.$$

For each fixed n, $0 \le x^n \le 1$ on $[0,1]$. Also, for each $x \in [0,1]$, the sequence x^n is monotone decreasing in n. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

converges by the alternating series test. Moreover, its partial sums are bounded.

By Abel's test for uniform convergence, since x^n is uniformly bounded and monotone in n, and since $\sum b_n$ converges, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

converges uniformly on $[0,1]$.

Equivalently, one may use the uniform alternating series test: for $x \in [0,1]$,

$$0 \le \frac{x^n}{n} \le \frac{1}{n},$$

and x^n/n decreases to 0 uniformly enough that the alternating remainders satisfy

$$|R_N(x)| \le \frac{x^{N+1}}{N+1} \le \frac{1}{N+1} \to 0.$$

Hence the convergence is uniform on $[0,1]$.

At $x = 1$, the value is

$$f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2,$$

which agrees with the continuous extension of $\ln(1+x)$ to $[0,1]$.

Final Answer

$$R = 1, \qquad f(x) = \ln(1+x) \text{ for } -1 < x < 1, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n \text{ converges uniformly on } [0,1].$$

End of Exam