Question	IA	IB	IIA	IIB	IIIA	IIIB	IIIC	IVA	IVB,C	V	VI(i-vi)	VI(vii)	**Total**
Mark													

Question I: [2+3= 5 points]

A. For the regular curve $\alpha: R \to R^3$ defined by:

$$\alpha(t) = \left(2t, t^2, \frac{t^3}{3}\right)$$

Find the arc length parameter (arc length function $s = f(t)$ only).

B. Find the curvature τ of the curve α in part **A.**

Question II: [3.5+1.5=5 points]

A. For the unit speed curve $\alpha(s)$ with

$$T(s) = \left(\frac{1}{2}\sqrt{1-s}, \frac{1}{2}\sqrt{1+s}, \frac{1}{\sqrt{2}}\right), \quad |s| < 1.$$

Compute k, N, B and τ for α.

B. Is the curve α in **A** a plane curve? Is α a helix? Justify your answer.

Question III: [3+1+2=6 points]

A. For the unit speed curve defined by

$$\alpha(s) = \left(\sin s\,, \frac{1}{\sqrt{2}}\cos s\,, \frac{1}{\sqrt{2}}\cos s \right).$$

Find the equation of the normal plane and deduce that α lies on a sphere by showing that the normal plane to α goes through a fixed point $x_0 \in R^3$.

B. When do we say that a curve β is an involute of a curve α?

C. Find an involute β of the unit speed curve $\alpha(s) = \left(\sin s\,, \frac{1}{\sqrt{2}}\cos s\,, \frac{1}{\sqrt{2}}\cos s \right)$.

Question IV: [1.5+2+2=5.5 points]

A. Let $x: R^2 \to R^3$ by be defined by

$$x(s,t) = (s, t^2, t^3).$$

Is x a simple surface? Justify your answer.

B. For any unit speed curve $\gamma(s) = x(\gamma_1(s), \gamma_2(s))$ lying on the simple surface x, prove that the normal curvature is given by:

$$k_n = \sum_{i,j=1}^{2} L_{ij} \, (\gamma_i)' (\gamma_j)'.$$

C. Let $x: U \to R^3$ be a simple surface. Then prove the following:

$$g = \|x_1 \times x_2\|^2,$$

where $g = det(g_{ij})$

Question V: [6 points]

Let $x: R^2 \to R^3$ be the simple surface defined by

$$x(u, v) = (u + v, u - v, u^2 - v^2),$$

and $\gamma: R \to R^3$, be defined by $\gamma(t) = x(t, 0)$. Then answer the following:

(i) Compute the geodesic curvature k_g. Is γ a geodesic? Justify your answer.

(ii) Compute the normal curvature k_n.

(iii) Compute the curvature k of γ.

Question VI: [12.5 points]

Let $x: R^2 \to R^3$ be the Monge patch defined by

$$x(u, v) = (u, v, u^2 + v^2).$$

Then answer the following:

(i) Compute the coefficients of the First Fundamental Form (metric coefficients $g_{ij}'s$).

(ii) Compute the Christoffel symbol Γ_{11}^1 and Γ_{12}^1 .

(iii) Compute the coefficients of the Second Fundamental Form ($L_{ij}'s$).

(iv) Find the Weingarten map L.

(v) Compute the mean curvature. Is x a minimal surface?

(vi) Compute the Gaussian curvature K.

(vii) Show that all points of x are elliptic.

Good Luck 😊