Model answers for the second midterm exam
Math 380 (Stochastic processes)
2^{nd} semester 2024/2025

Problem 1:

1) $P(X_0=0, X_1=1, X_2=2) = P_0 P_{01} P_{12} = (0.2)(0.2)(0.7) = 0.028$

2) $$P(X_2=1, X_3=1 \mid X_1=0) = \frac{P(X_1=0, X_2=1, X_3=1)}{P(X_1=0)}$$

$$= \frac{\cancel{P(X_1=0)} P_{01} P_{11}}{\cancel{P(X_1=0)}} = P_{01} P_{11} = (0.2)(0.3) = 0.06$$

4

$$P(X_1=1, X_2=1 \mid X_0=0) = \frac{P(X_0=0, X_1=1, X_2=1)}{P(X_0=0)} = \frac{\cancel{P_0} P_{01} P_{11}}{\cancel{P_0}} = (0.2)(0.3) = 0.06$$

3) we have $X_0=2$ known, so $P(X_0=2)=1$ (certain)

$$P_2 = 1, \quad P_0 = P_1 = 0$$

$$P(X_0=2, X_1=1, X_2=0) = P(X_0=2) P_{21} P_{10} = (1)(0)(0) = 0$$

$$P(X_2=1) = P_0 P_{01}^{(2)} + P_1 P_{11}^{(2)} + P_2 P_{21}^{(2)} = (0) P_{01}^{(2)} + (0) P_{11}^{(2)} + (1) P_{21}^{(2)}$$

$$= P_{21}^{(2)} = (0.4)(0.2) + (0)(0.3) + (0.6)(0) = 0.08$$

Problem 2:

1) $$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$$

2) $$P(X_1=0, X_2=0, X_3=0, X_4=1 \mid X_0=0) = \frac{P(X_0=0, X_1=0, X_2=0, X_3=0, X_4=1)}{P(X_0=0)}$$

3

$$= \frac{\cancel{P_0} P_{00} P_{00} P_{00} P_{01}}{\cancel{P_0}}$$

$$= P_{00}^3 P_{01} = (0.8)^3 (0.2)$$

$$= 0.1024$$

3) $P(X_3=1 \mid X_0=1) = P_{11}^{(3)} = 0.376$

as $P^2 = \begin{pmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{pmatrix}$ and $P^3 = \begin{pmatrix} 0.688 & 0.312 \\ 0.624 & 0.376 \end{pmatrix}$

problem 3:

player's I fortune is $i=10$ and the total of fortunes is $N=10+20=30$. The craps is $p=0.494$ and $q=1-p=1-0.494=0.506$. Thus, by the formula we obtain:

3

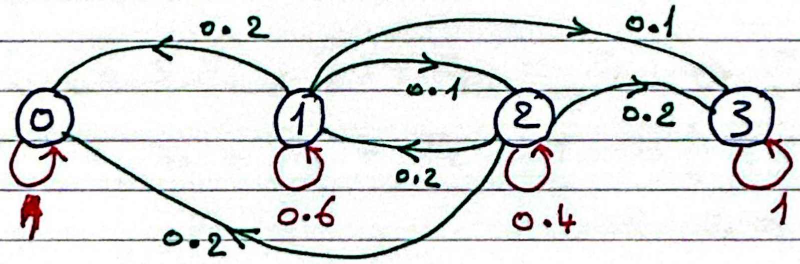
$$u_{10}=P\{X_n=0|X_0=5\}=\frac{\left(\frac{0.506}{0.494}\right)^{10}-\left(\frac{0.506}{0.494}\right)^{30}}{1-\left(\frac{0.506}{0.494}\right)^{30}}\approx 0.74275$$

problem 4:

1) $\boxed{\alpha=1}$, $0.2+\beta+0.1+0.1=1 \Rightarrow \boxed{\beta=0.6}$

$\gamma+0.2+0.4+0.2=1 \Rightarrow \boxed{\gamma=0.2}$

2)



5

0 and 3 are absorbing states.
1 and 2 are transient states.

3)

put $u_1=P\{X_T=0|X_0=1\}$ and $u_2=\{X_T=0|X_0=2\}$

The first step analysis yields the equations:

$$\text{and}\begin{cases}u_1=0.2+0.6u_1+0.1u_2\\ u_2=0.2+0.2u_1+0.4u_2\end{cases}\Leftrightarrow\begin{cases}4u_1-u_2=2\\ -2u_1+6u_2=2\end{cases}$$

$$\Rightarrow \boxed{u_1=\frac{7}{11}} \text{ and } u_2=\frac{6}{11}$$

Also, if $v_1=E[T|X_0=1]$ and $v_2=E[T|X_0=2]$, then

$$\begin{cases}v_1=1+0.6v_1+0.1v_2\\ v_2=1+0.2v_1+0.4v_2\end{cases}\Rightarrow\begin{cases}4v_1-v_2=10\\ -2v_1+6v_2=10\end{cases}\Rightarrow\begin{cases}\boxed{v_1=\frac{35}{11}}\\ v_2=\frac{30}{11}\end{cases}$$

problem 5 :

1) Consider the states : $\begin{cases} 0 & \text{poor Customer} \\ 1 & \text{Satisfactory Customer} \\ 2 & \text{preferred Customer} \end{cases}$

So the transition probability matrix is :

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

2) The equations determining (π_0, π_1, π_2) are

$$\begin{cases} 0.6\,\pi_0 + 0.1\,\pi_1 = \pi_0 \implies \pi_0 = \frac{1}{4}\pi_1 \\ \cancel{0.4\,\pi_0 + 0.6\,\pi_1 + 0.2\,\pi_2 = \pi_1} \leftarrow \text{strike this one} \\ 0.3\,\pi_1 + 0.8\,\pi_2 = \pi_2 \implies \pi_2 = \frac{3}{2}\pi_1 \end{cases}$$

5

with $\pi_0 + \pi_1 + \pi_2 = 1 \implies \frac{1}{4}\pi_1 + \pi_1 + \frac{3}{2}\pi_1 = 1$

$$\implies \pi_1 = \frac{4}{11}$$

$$\implies \pi_2 = \frac{6}{11}, \ \pi_0 = \frac{1}{11}$$

poor drivers $\longrightarrow \pi_0 = \frac{1}{11}$

Satisfactory drivers $\longrightarrow \pi_1 = \frac{4}{11}$

preferred drivers $\longrightarrow \pi_2 = \frac{6}{11}$

problem 6 :

1) We label the states as follows :

state (i,i) if the stock increased both today and yesterday

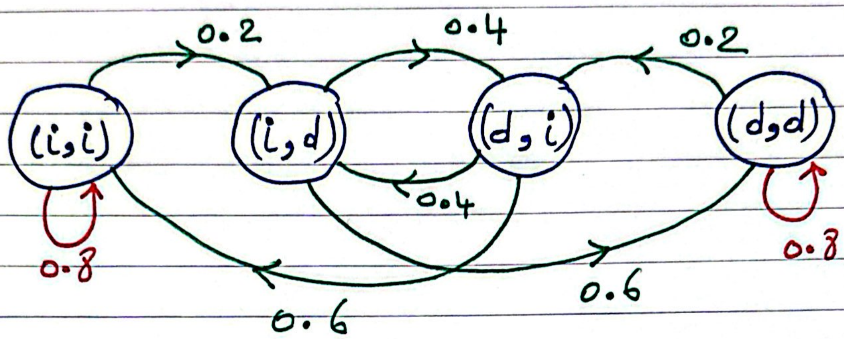
state (i,d) if the stock increased yesterday and decreased today

state (d,i) if the stock decreased yesterday and increased today

state (d,d) if the stock decreased both yesterday and today

So the transition probability matrix reads as :

$$
\begin{array}{c|cccc}
 & (i,i) & (i,d) & (d,i) & (d,d) \\
\hline
(i,i) & & & & \\
(i,d) & & & & \\
(d,i) & & & & \\
(d,d) & & & &
\end{array}
\qquad
P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}
$$



2) The equations determining $(\pi_0, \pi_1, \pi_2, \pi_3)$ are

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$$
\begin{cases}
0.8\pi_0 + 0.6\pi_2 = \pi_0 \Rightarrow \pi_2 = \frac{1}{3}\pi_0 \\
0.2\pi_0 + 0.4\pi_2 = \pi_1 \Rightarrow \pi_1 = \frac{1}{5}\pi_0 + \frac{2}{5}\pi_2 = \frac{1}{3}\pi_0 \\
\cancel{0.4\pi_1 + 0.2\pi_3 = \pi_2} \ \text{strike this one} \\
0.6\pi_1 + 0.8\pi_3 = \pi_3 \Rightarrow \pi_3 = 3\pi_1 = \pi_0 \\
\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \pi_0 + \frac{1}{3}\pi_0 + \frac{1}{3}\pi_0 + \pi_0 = 1
\end{cases}
$$

$$\Rightarrow \pi_0 = \frac{3}{8}$$

$$\Rightarrow (\pi_0, \pi_1, \pi_2, \pi_3) = \left(\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}\right)$$

So the fraction of days in which the stock increases is

$$\pi_0 + \pi_1 = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}.$$

3) By the above calculation, the fraction of days in which the stock decreases is :-

$$\pi_2 + \pi_3 = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}.$$