# Q1

(a) A stochastic process $\{X_n; n = 0,1,2,...\}$ is a martingale if for all $n \geq 1$,

(i) $E|X_n| < \infty$

(ii) $E[X_{n+1}|\mathcal{F}_n] = X_n$

(b)

For $n \geq 1$,

1- $E|X_n| = E|S_1 + S_2 + .... + S_n| \leq E|S_1| + E|S_2| + ..... + E|S_n| < \infty$,
since $E|S_i| < \infty$ for all i=1,2,…,n.

2- $E[X_{n+1}|\mathcal{F}_n] = E[(\overbrace{S_1 + S_2 + .... + S_n}^{X_n} + S_{n+1})|\mathcal{F}_n]$, → as $\{E[X_n|F_n] = X_n = S_1 + \dots + S_n$ because $X_n$ is $F_n$-adapted

$= (S_1 + S_2 + .... + S_n) + E[S_{n+1}|\mathcal{F}_n]$

$= X_n + E[S_{n+1}]$ → as $\{S_{n+1}$ independent of $F_n$

$= X_n$ if and only if $E[S_{n+1}] = 0$ .

Therefore, $X_n$ is a martingale if and only if $E[S_n] = 0$ for all $n \geq 1$.

## Q2 : (a) (i)

The fortune for player A is $i = \$5$ and the total amount is $N = \$5 + \$10 = \$15$

p=0.5071 $\Rightarrow$ q=0.4929

$u_i = pr\{X_n \text{ reaches state 0 before state } N | X_0 = i\}$

$$u_i = \frac{(q/p)^i - (q/p)^N}{1-(q/p)^N}, \; p \neq q$$

$$\therefore u_i = \frac{(0.4929/0.5071)^5 - (0.4929/0.5071)^{15}}{1-(0.4929/0.5071)^{15}}$$

$u_i = 0.61837$

(ii) if $p = 0.5 \Rightarrow q = 1 - p = 0.5$

So that $u_i = \frac{N-i}{N} = \frac{15-5}{15} = 0.667$

## Q2 : (b)

$$P = \begin{bmatrix} \overset{1-a}{0.99} & \overset{a}{0.01} \\ \underset{b}{0.12} & \underset{1-b}{0.88} \end{bmatrix} \xrightarrow{\text{long run}} \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

$$\downarrow \pi_0 \qquad \downarrow \pi_1$$

$$\Rightarrow \pi_0 = \frac{0.12}{0.01+0.12} = \frac{0.12}{0.13}$$

$$\& \; \pi_1 = \frac{0.01}{0.01+0.12} = \frac{0.01}{0.13} = \frac{1}{13}$$

as $C_0 = 4\$$ & $C_1 = 7\$$

$\Rightarrow$ the long run mean cost per period

$$= 4 * \frac{12}{13} + 7 * \frac{1}{13} = \underline{4.2308} \, \$$$

Q3: a)

$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n \mid X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$

$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n}$ Definition of Markov

By repeating this argument $n-1$ times

$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$

$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

Q3:

b) $P_{GG} = P_r\{X_{n+1} = G \mid X_n = G\} = 0.4$

$P_{GD} = P_r\{X_{n+1} = D \mid X_n = G\} = 0.6$

$P_{DD} = P_r\{X_{n+1} = D \mid X_n = D\} = 0.3$

$P_{DG} = P_r\{X_{n+1} = G \mid X_n = D\} = 0.7$

$P_r\{X_2 = G, X_3 = G, X_4 = G, X_5 = G, X_6 = D \mid X_1 = G\} = ?$

conditional Property

$= P_r\{X_6 = D \mid X_2 = G, X_3 = G, X_4 = G, X_5 = G, X_1 = G\} \cdot P_r\{X_2 = G, X_3 = G, X_4 = G, X_5 = G \mid X_1 = G\}$

↓ Markov property

$= P_r\{X_6 = D \mid X_5 = G\} \cdot P_r\{X_2 = G, X_3 = G, X_4 = G, X_5 = G \mid X_1 = G\}$

$= P_{GD}$ . repeat the step.

$\left\{ P_r\{X_5 = G \mid X_2 = G, X_3 = G, X_4 = G, X_1 = G\} \cdot P_r\left\{ \begin{matrix} X_2 = G \\ X_3 = G \\ X_4 = G \end{matrix} \middle| X_1 = G \right\} \right\}$

↓

Markov Property

$P_r\{X_5 = G \mid X_4 = G\}$

$= P_{GD} \cdot P_{GG} \cdot \ldots$ and so on continous.

we get:

$= P_{GD} \cdot P_{GG}^4 = 0.6 (0.4)^4 = 0.01536$

## Q4:

$u_i = pr\{X_T = 0 | X_0 = i\}$ for i=1,2,

*and* $v_i = E[T | X_0 = i]$ for i=1,2.

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.1 + 0.6u_1 + 0.1u_2$$
$$u_2 = 0.2 + 0.3u_1 + 0.4u_2$$

$\Rightarrow$

$$4u_1 - u_2 = 1 \qquad (1)$$
$$3u_1 - 6u_2 = -2 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{8}{21} \text{ and } u_2 = \tfrac{11}{21}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \tfrac{11}{21}$$
$$\approx 0.52$$

Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.6v_1 + 0.1v_2$$
$$v_2 = 1 + 0.3v_1 + 0.4v_2$$

$\Rightarrow$

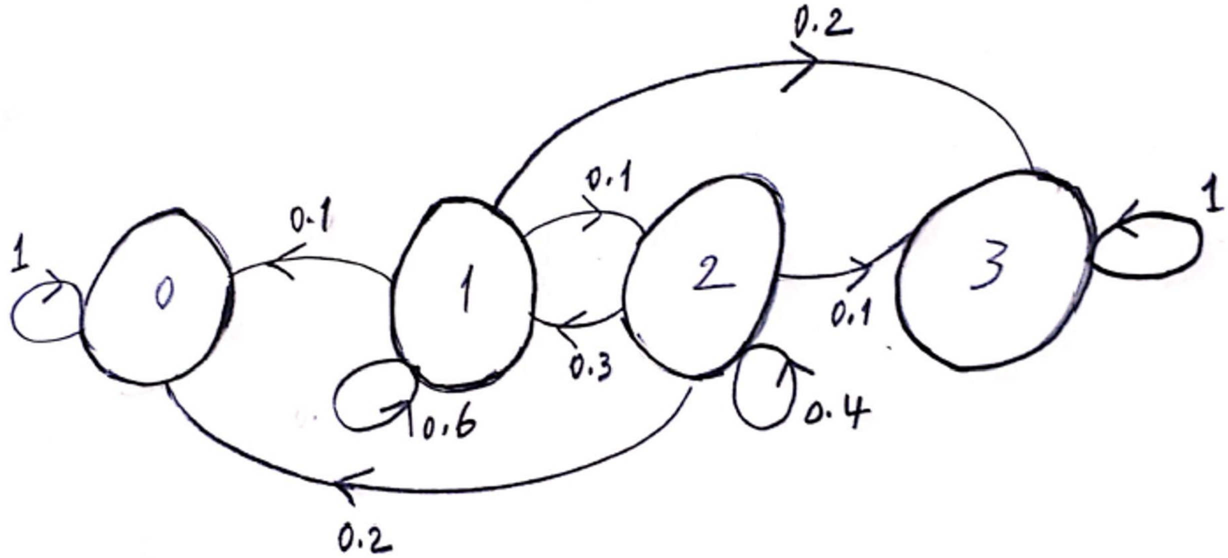
$$4v_1 - v_2 = 10 \qquad (1)$$
$$3v_1 - 6v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

∴The mean time to absorption is $v_1 = v_2 = \tfrac{10}{3}$

$$\therefore\ v_2 = v_{20} = \tfrac{10}{3}$$
$$\approx 3.3$$

And It's an absorbing Markov Chain,




**Markov Chain Diagram**

## Q5 :(a)

The Markov chain $X_0, X_1, X_2, \ldots$ represents the day's weather

$\because\ pr(X_0 = 1) = p_1 \ = 3/8$

$\therefore\ pr(X_0 = 2) = p_2 \ = 5/8$

$\Rightarrow$ The initial probability distribution is $[3/8 \quad 5/8]$

To get the prob. of weather will be rainy on 2nd June:

$$\begin{aligned} pr(X_1 = 1) &= \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2) \\ &= P_{11}p_1 + P_{21}p_2 \\ &= (0.8)(\tfrac{3}{8}) + (0.4)(\tfrac{5}{8}) \end{aligned}$$

$\therefore pr(X_1 = 1) = 0.55$

To get the prob. of weather will be rainy on 3rd June:

$$\because\ \mathbf{P}^2 = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{bmatrix}$$

$$\begin{aligned} pr(X_2 = 1) &= \Pr(X_2 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_2 = 1 | X_0 = 2)\Pr(X_0 = 2) \\ &= P_{11}{}^2 p_1 + P_{21}{}^2 p_2 \\ &= (0.72)(3/8) + (0.56)(5/8) \end{aligned}$$

$\therefore pr(X_2 = 1) = 0.62$

Q5:(b)

The transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.4 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0.4 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \end{array}$$

where,

$$P_{ij} = \Pr\left\{X_{n+1} = j \middle| X_n = i\right\}$$

$$= \begin{cases} \Pr(\xi_{n+1} = 3 - j), \ i \le 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), \ 0 < i \le 3 & \text{without replenishment} \end{cases}$$