Problem 1 [6 marks]:
Consider the following matrix:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0.2 & \beta & 0.3 & 0.4 \\ 0.3 & 0.2 & \gamma & 0.3 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \end{array}$$

1. Find α, β, γ, λ, such that $\mathbf{P}$ defines a transition probability matrix of some Markov chain $\{X_t\}$.
2. Assuming the values of $\alpha, \beta, \gamma, \lambda$ obtained in (1), sketch its transition diagram, and classify its states (as absorbing or communicating).
3. Determine the conditional probability $\mathbb{P}\{X_1 = 2, X_2 = 2 | X_0 = 1\}$.
4. Starting in state 1, determine the probability that this process ends in state 3.
5. Determine the mean time to absorption.

Problem 2 [6 marks]:
An auto insurance company classifies its customers in three categories: poor (labelled state 0), satisfactory (labelled state 1) and preferred (labelled state 2). Assume the following happens within one year:

- No customer is upgraded from poor to preferred category, or is downgraded from preferred to poor category.
- 40% of poor category become satisfactory.
- 25% of satisfactory customers move to preferred category.
- 10% of satisfactory customers become poor.
- 25% of preferred customers are downgraded to satisfactory.

1. Write the transition probability matrix associated to this model.
2. What is the long run fraction of drivers in each of these three categories.
3. Suppose that every period that this process spends in state 0 incurs a cost of \$270000. Every period that the process spends in state 1 incurs a cost of \$180000. Every period that the process spends in state 2 incurs a cost of \$90000. What is the long run mean cost per period associated with this Markov chain?

!!! Please turn the page over !!!

Problem 3 [7 marks]:
The claims department of an insurance company receives envelopes with claims for insurance coverage at a Poisson rate of $\lambda = 60$ envelopes per week. Assume that, for any period of time, the number of envelopes $N(t)$ and the numbers of claims in the envelopes X_i are independent, and that the numbers of claims in the envelopes have the following discrete distribution X:

Number of claims	Probability
1	0.30
2	0.15
3	0.20
4	0.35

Assuming that the total number of claims evolves as a compound Poisson process approximated by a normal distribution with mean $\mu = \mathbb{E}[N]\mathbb{E}[X]$ and variance $\sigma^2 = \mathbb{E}[N]\mathbb{E}[X^2]$, calculate the 75^{th} percentile of the number of claims received in 12 weeks.

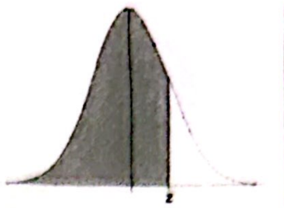
Problem 4 [6 marks]:
Suppose that the value of currencies in two countries **A** and **B** is currently the same, (in other words: 1 unit in country **A** can be exchanged for 1 unit in country **B**). Denote by $D(t)$ the difference between the currency values in countries **A** and **B** at any moment of time, (in other words: 1 unit in country **A** will exchange for $1 + D(t)$ at time t in country **B**). Assume that the difference $D(t)$ is modelled as a Brownian motion with drift 0 and variance parameter 0.04. An investor in country **A** currently invests 1 in a risk free investment in country **B** that matures at 1.75 units in the currency of country **B** in 5 years. After the first year, 1 unit in country **A** is worth 1.08 in country **B**. Calculate the conditional probability after the first year that when the investment matures and the funds are exchanged back to country **A**, the investor will receive at least 1.75 in the currency of country **A**.

Problem 5 [15 marks]:

1. Let $\{X(t),\ t \geq 0\}$ be a Brownian motion with variance parameter $\sigma^2 = 8$. Calculate the following:
   (a) $\mathbb{P}\{|X(4) - X(2)| \geq 10\}$.
   (b) $\mathbf{Var}\,(3 + X(4) - 2X(2) + X(3))$.
   (c) $\mathbb{P}\{X(2) - 2X(3) \leq 4\}$.
   (d) $\mathbf{Cov}\,(3 + X(4) - 2X(2)\ ,\ 5 - X(3))$.
2. Let $B(t)$ be a Brownian motion. Show that the process $X(t) = B(T - t) - B(T)$ is also a Brownian motion on $[0, T]$, $T < \infty$.
3. Let $B(t)$ be the standard Brownian motion. Show that $B(t)$ is a martingale.
4. Let $B(t)$ be the standard Brownian motion. Determine the value of the parameter α such that the process $X(t) = e^{B(t) - \alpha t}$, $t \geq 0$, defines a martingale.

Good Luck

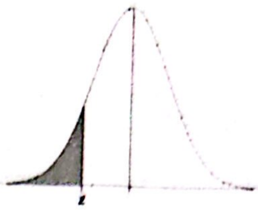
Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Standard Normal Cumulative Probability Table



Cumulative probabilities for NEGATIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2511	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Math 380, stochastic processes
Model answers for Final Exam
1st semester 1447 H, 2025/2026

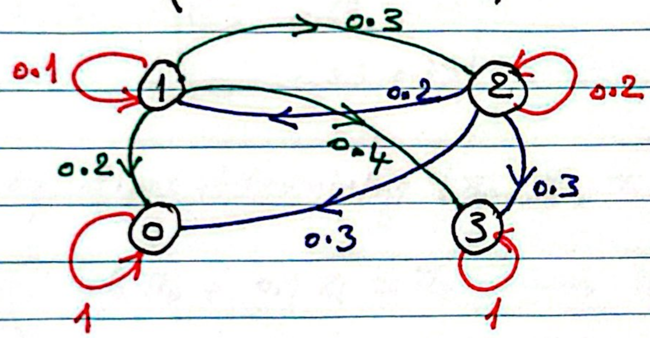
problem 1.-

1) $\boxed{\alpha = 1}$, $0.2 + \beta + 0.3 + 0.4 = 1 \Rightarrow \boxed{\beta = 0.1}$

$0.3 + 0.2 + \gamma + 0.3 = 1 \Rightarrow \boxed{\gamma = 0.2}$, $\boxed{\lambda = 1}$

Then, the transition matrix becomes:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.1 & 0.3 & 0.4 \\ 0.3 & 0.2 & 0.2 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

2) 

6

3) $P\{X_1 = 2, X_2 = 2 \mid X_0 = 1\} = \dfrac{P(X_0 = 1, X_1 = 2, X_2 = 2)}{P(X_0 = 1)}$

$$= \frac{\cancel{P_1}\, P_{12} P_{22}}{\cancel{P_1}} = P_{12} P_{22}$$

$$= (0.3)(0.2) = \boxed{0.06}$$

4) $u_1 = P\{X_T = 3 \mid X_0 = 1\}$ and $u_2 = P\{X_T = 3 \mid X_0 = 2\}$

The first step analysis yields the equations

$$\begin{cases} u_1 = 0.1\,u_1 + 0.3\,u_2 + 0.4 \\ u_2 = 0.2\,u_1 + 0.2\,u_2 + 0.3 \end{cases} \Leftrightarrow \begin{cases} 9u_1 - 3u_2 = 4 \\ -2u_1 + 8u_2 = 3 \end{cases}$$

Solving the system, we obtain

$$\Rightarrow u_2 = \frac{35}{66}, \quad \boxed{u_1 = \frac{41}{66}}$$

5) $\vartheta_1 = E[T \mid X_0 = 1]$ and $\vartheta_2 = E[T \mid X_0 = 2]$, Then

$$\begin{cases} \vartheta_1 = 1 + 0.1\,\vartheta_1 + 0.3\,\vartheta_2 \\ \vartheta_2 = 1 + 0.2\,\vartheta_1 + 0.2\,\vartheta_2 \end{cases} \Rightarrow \begin{cases} 9\,\vartheta_1 - 3\,\vartheta_2 = 10 \\ -2\,\vartheta_1 + 8\,\vartheta_2 = 10 \end{cases}$$

solving this system, we obtain

$$\vartheta_2 = \frac{110}{66} = \frac{5}{3}, \quad \boxed{\vartheta_1 = \frac{5}{3}}.$$

problem 2 :

6

1) The transition probability matrix is :

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.65 & 0.25 \\ 0 & 0.25 & 0.75 \end{pmatrix} \end{array}$$

2) The equations determining (π_0, π_1, π_2) are

$$\begin{cases} 0.6\,\pi_0 + 0.1\,\pi_1 = \pi_0 \longrightarrow \pi_0 = \frac{1}{4}\pi_1 \\ \cancel{0.4\,\pi_0 + 0.65\,\pi_1 + 0.25\,\pi_2 = \pi_1} \quad \text{strike this equation} \\ 0.25\,\pi_1 + 0.75\,\pi_2 = \pi_2 \longrightarrow \pi_2 = \pi_1 \\ \pi_0 + \pi_1 + \pi_2 = 1 \longrightarrow \frac{1}{4}\pi_1 + \pi_1 + \pi_1 = 1 \end{cases}$$

$$\Rightarrow \pi_1 = \frac{4}{9}, \quad \pi_2 = \frac{4}{9}, \quad \pi_0 = \frac{1}{9}$$

poor drivers $\longrightarrow \boxed{\pi_0 = \frac{1}{9}}$

satisfactory drivers $\longrightarrow \boxed{\pi_1 = \frac{4}{9}}$

preferred drivers $\longrightarrow \boxed{\pi_2 = \frac{4}{9}}$

3) The long run mean cost per period $= \sum_{j=0}^{2} \pi_j C_j$

$$\Rightarrow \text{Cost} = (270\,000)\left(\frac{1}{9}\right) + (180\,000)\left(\frac{4}{9}\right) + (90\,000)\left(\frac{4}{9}\right) = 30\,000 + 80\,000 + 40\,000$$

$$= 150\,000 \ \$.$$

problem 3:

The total number of claims recieved in 12 weeks is $Y = \sum_{i=1}^{N(12)} X_i$.

we have

$$E[X] = (1)(0.3) + (2)(0.15) + (3)(0.2) + (4)(0.35) = 2.6$$

$$E[X^2] = (1)^2(0.3) + (2)^2(0.15) + (3)^2(0.2) + (4)^2(0.35) = 8.3$$

$$\mu = E[Y] = E[N(12)].E[X] = (60)(12)(2.6) = 1872$$

$$\sigma^2 = Var[Y] = E[N(12)] \cdot E[X^2] = (60)(12)(8.3) = 5976$$

7

If q is the 75^{th} percentile of the number of claims received in 12 weeks, then we have

$$0.75 = P(Y \leqslant q) \simeq P\left(N(0,1) \leqslant \frac{q-\mu}{\sigma}\right) = P\left(N(0,1) \leqslant \frac{q-1872}{\sqrt{5976}}\right)$$

From the normal table, we see that $0.75 = P(N(0,1) \leqslant 0.68)$

$$\Rightarrow \frac{q-1872}{\sqrt{5976}} = 0.68 \Rightarrow q = (0.68)(77.304) + 1872 = 52.56 + 1872$$

$$\Rightarrow q = 1924.567 \longrightarrow \boxed{q \approx 1924}$$

problem 4:

By the Formula (See Definition 5.18 in Lecture Notes):

$$D(t) = \underbrace{\sigma}_{\sqrt{0.04}} B(t) + \underbrace{\mu}_{0} t, \quad B(t) \text{ is a Standard Brownian motion.}$$

$$\Rightarrow D(t) = 0.2\, B(t)$$

6

$$P\left\{\frac{\cancel{1.75}}{D(5)} > \cancel{1.75} \mid D(1) = 1.08\right\} = P\{D(5) < 1 \mid D(1) = 1.08\}$$

Exchange $1 + B(t)$ from A to B

$$= P\{1 + 0.2B(5) < 1 \mid 1 + 0.2B(1) = 1.08\}$$

$$= P\{B(5) < 0 \mid B(1) = 0.4\} = P\{\underbrace{B(5) - B(1) < -0.4 \mid B(1)}_{\text{independent}} = 0.4\}$$

$$= \phi\left(\frac{-0.4}{\sqrt{4}}\right) = P\{B(5) - B(1) < -0.4\} \rightarrow B(5) - B(1) \sim N(0,4),$$

$$= \underbrace{P\{N(0,1) \leqslant -0.2\}}_{\phi(-0.2)} = \boxed{0.4207}$$

↑normal table

2) a) $X(t)-X(s)\sim N(0,t-s)$, $0\leqslant s\leqslant t<\infty$? , $(T-s\geqslant T-t)$

$X(t)-X(s)=B(T-t)-\cancel{B(T)}-B(T-s)+\cancel{B(T)}=-(B(T-s)-B(T-t))$

$\sim -N(0,\cancel{T}-s-\cancel{T}+t)=N(0,(-1)^2(t-s))=N(0,t-s)$.

b) $Cov(X(t),X(s))=\min(s,t)$? Let $s<t$, then

$Cov(X(t),X(s))=Cov(B(T-t)-B(T),B(T-s)-B(T))$

3

$=Cov(B(T-t),B(T-s))-Cov(B(T-t),B(T))-Cov(B(T),B(T-s))+Cov(B(T),B(T))$

$=\min\{T-t,T-s\}-\min\{T-t,T\}-\min\{T,T-s\}+\min\{T,T\}$

$=T-t-(T-t)-(T-s)+T=\cancel{T}-\cancel{t}-\cancel{T}+\cancel{t}-\cancel{T}+s+\cancel{T}=s=\min\{s,t\}$

A similar calculation for the case $s>t$.

c) $E[X(t)]=E[B(T-t)-B(T)]=E[B(T-t)]-E[B(T)]=0$.

3) a) $E[|X(t)|]<\infty$?

$X(t)=B(t)$ and we know that $|B(t)|\leqslant 1+B^2(t)$, $E[B^2(t)]=t$

So we get $E[|B(t)|]\leqslant E[1+B^2(t)]\leqslant 1+t<\infty$.

b) $E[X(t)|F_s]=X(s)$ for $0\leqslant s<t$ and $\{F_s\}$ a filtration?

3

$E[B(t)|F_s]=E[B(t)-B(s)+B(s)|F_s]$

$=E[B(t)-B(s)|F_s]+E[B(s)|F_s]$

$=\underbrace{E[B(t)-B(s)]}_{=0}+B(s)=B(s)$.

$\Rightarrow B(t)$ is a martingale.

4) suppose that $X(t)=e^{B(t)-\alpha t}$, $t\geqslant 0$ is a martingale. Then

$E[e^{B(t)-\alpha t}|F_s]=e^{B(s)-\alpha s}$, $\{F_s\}$ a filtration.

$\Rightarrow E[e^{B(t)-B(s)+B(s)-\alpha t}|F_s]=E[e^{B(t)-B(s)}e^{B(s)-\alpha t}|F_s]$

$=e^{B(s)-\alpha t}E[e^{B(t)-B(s)}|F_s]=e^{B(s)-\alpha t}E[e^{B(t)-B(s)}]$ $=e^{\frac{t-s}{2}}$

3

$=e^{B(s)-\alpha t+\frac{t-s}{2}}=e^{B(s)-\alpha s}\Rightarrow \cancel{B(s)}-\alpha t+\frac{t-s}{2}=\cancel{B(s)}-\alpha s$

$\Rightarrow \alpha(t-s)=\frac{1}{2}(t-s)\Rightarrow \boxed{\alpha=\frac{1}{2}}$.

problem 5:

1) a) $P\{|X(4)-X(2)| \geq 10\} = P\{X(4)-X(2) \geq 10 \text{ or } X(4)-X(2) \leq -10\}$

$$= 1 - P(X(4)-X(2) \leq 10) + P(X(4)-X(2) \leq -10\}$$

$$= 1 - P\left(B(4)-B(2) \leq \frac{10}{\sqrt{8}}\right) + P\left(B(4)-B(2) \leq -\frac{10}{\sqrt{8}}\right)$$

$$= 1 - P\left(Z \leq \frac{10}{4}\right) + P\left(Z \leq -\frac{10}{4}\right)$$

$$= 1 - 0.9938 + 0.0062 = 2(0.0062)$$

$$= \boxed{0.0124.}$$

b) $Var(3 + X(4) - 2X(2) + X(3)) = Var(3 + (X(4)-X(3)) + 2(X(3)-X(2)))$

$$= Var(X(4)-X(3)) + (2)^2 \, Var(X(3)-X(2))$$

$$= Var(\sqrt{8}(B(4)-B(3))) + 4\,Var(\sqrt{8}(B(3)-B(2))) = 8\,Var(N(0,1)) + 4 \times 8\,Var(N(0,1))$$

$$= (8)(1) + (4)(8)(1) = \boxed{40}$$, as $B(4)-B(3) \sim N(0, 4-3) = N(0,1)$, $B(3)-B(2) \sim N(0, 3-2) = N(0,1)$

15

c) $X(2) - 2X(3)$ *has a normal distribution with mean and variance*

$$Var(X(2) - 2X(3)) = Var(\sqrt{8}B(2) - 2\sqrt{8}B(3))$$

$$= Var(-\sqrt{8}B(2) - 2\sqrt{8}(B(3)-B(2)))$$

6

$$= (-\sqrt{8})^2(2) + (-2\sqrt{8})^2(3-2)$$

$$= 16 + 32 = 48.$$

Hence $P(X(2) - 2X(3) \leq 4) = P\left(Z \leq \frac{4}{\sqrt{48}}\right) = P(Z \leq 0.577) = \boxed{0.7157}$

d) $Cov(3 + X(4) - 2X(2), 5 - X(3)) = Cov(X(4) - 2X(2), -X(3))$

$$= Cov(X(4), -X(3)) + Cov(-2X(2), -X(3))$$

$$= -Cov(X(4), X(3)) + (-2)(-1)\,Cov(X(2), X(3))$$

$$= -8\min(4,3) + 2 \times 8 \min(2,3)$$

$$= -24 + 32 = \boxed{8}$$

as $Cov(B(t), B(s)) = \min\{t, s\}$, and $X(t) = \sqrt{8}B(t)$.