King Saud University
College of Sciences
Department of Mathematics

Second Mid-Term Exam, S2 1445
Math 380 – Stochastic Processes
Time: 2 hours

Answer all questions:

Q1: [1+4]

(a) Define a martingale.

(b) Let $S_1, S_2, \ldots. S_n$ be independent random variables such that $E[|S_i|] < \infty$, for all $i = 1, 2, 3, \ldots., n$. Let $X_0 = 0$, $X_n = S_1 + S_2 + \ldots. + S_n$, $n \geq 1$.

Prove that, X_n is a martingale if and only if $E[S_n] = 0$, for all $n \geq 1$.

Q2: [2 + 1 +2]

(a) The probability of the thrower winning in the dice game is $p = 0.5071$. Suppose player A is the thrower and begins the game with \$5, and player B, his opponent, begins with \$10.

(i) What is the probability that player A goes bankrupt before player B? Assuming that the bet is \$1 per round.

(ii) What if $p = 0.5$, does the probability that player A goes bankrupt before player B change or stays the same, justify your answer.

(b) Let X_n denote the quality of the n^{th} item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that X_n evolves as a Markov chain whose transition matrix is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{array}$$

Assume that every period that the process spends in state 0 incurs a cost \$4, while every period that the process spends in state 1 incurs a cost of \$7. What is the long run mean cost per period associated with this Markov chain?

Q3: [2+3]

(a) For the Markov process $\{X_t\}$, $t = 0,1,2,...,n$ with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$. Prove that $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$, where $p_{i_0} = \Pr\{X_0 = i_0\}$.

(b) Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability 0.4, and is followed by a defective item with probability 0.6. Similarly, a defective item is followed by another defective item with probability 0.3, and is followed by a good item with probability 0.7. If the first item is good, what is the probability that the first defective item to appear is the sixth item?

Q4: [1+2+2]

Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{Vmatrix} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \end{Vmatrix}$$

(i) Sketch the Markov chain diagram, and determine whether it is an absorbing chain or not.

(ii) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(iii) Determine the mean time to absorption.

Q5: [2 + 3]

(a) For modelling weather phenomenon, let $\{X_n\}$ be a Markov chain with state space $S=\{1, 2\}$, where 1 stands for rainy and 2 stands for dry. The transition probability matrix is given by:

$$P = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{Vmatrix} \begin{array}{cc} 1 & 2 \\ 0.8 & 0.2 \\ 0.4 & 0.6 \end{array} \end{Vmatrix}$$

Initially, assume that the probability of weather will be rainy on 1st June equals $\frac{3}{8}$.

Find each of the following:

(i) The initial probability distribution. (*Hint: find them from the given information*).

(ii) The probability that the weather will be rainy on 2nd June.

(iii) The probability that the weather will be rainy on 3rd June.

(b) Consider a spare parts inventory model in which either 0, 1, 2 or 3 repair parts are demanded in any period, with

$Pr\{\xi_n = 0\} = 0.1, \quad Pr\{\xi_n = 1\} = 0.4, Pr\{\xi_n = 2\} = 0.3, Pr\{\xi_n = 3\} = 0.2,$

and suppose that $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.
