King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1446
Math 380 – Stochastic Processes
Time: 3 hours – **Marks: 40**

Answer all questions

Question 1: 6[3+3]

(a) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$P = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0.5 & 0 & 0.5 \end{array} \right\| \end{array}$$

Determine the probabilities:

(i) $Pr\{X_2 = 1, X_3 = 1 | \, X_1 = 0\}$

(ii) $Pr\{\, X_3 = 1 | \, X_0 = 0 \,\}$

(b) Let $\zeta_1, \zeta_2, \ldots.$ be independent Bernoulli random variables with parameter p,

$0 < p < 1$. Show that: $X_0 = 1$ and $X_n = p^{-n}\, \zeta_1 \ldots \ldots \zeta_n,\ n = 1,2, \ldots.,$ defines a non-negative martingale.

Question 2: 8[5+3]

(a) Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with a probability of 0.5, if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with a probability of 0.4, if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.7, if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.2.

(i) Transform this model into a Markov chain, and then find the transition probability matrix.

(ii) Find the long-run fraction of days in which it is cloudy, and also the long-run fraction of days in which it is sunny.

(b) Find the mean time to reach state 3 starting from state 0 for the Markov chain whose transition probability matrix is:

$$P = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \left\| \begin{array}{cccc} 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0.7 & 0.1 & 0.2 \\ 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

Question 3: 8[4+4]

(a) Using the differential equations

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda p_{n-1}(t) - \lambda p_n(t), \;\; n = 1, 2, 3, \ldots \qquad (2)$$

where all birth parameters are the same constant λ with initial condition $X(0) = 0$.

Show that

$$p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \;\; n = 0, 1, 2, \ldots$$

(b) A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0$, $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 5$. Determine $P_n(t)$ for $n = 1, 2, 3$.

Question 4: 8[4+2+2]

(a) Suppose that customers arrive at a facility according to a Poisson process having a rate $\lambda = 3$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities:

(i) $Pr\{X(2) = 3\}$.

(ii) $Pr\{X(2) = 3 \; and \; X(4) = 7\}$.

(iii) $Pr\{X(4) = 7 | X(2) = 3\}$.

(b) Suppose that customers arrive at a clinic according to a non-homogenous Poisson process having the rate function:

$$\lambda(t) = \begin{cases} 2t+1, & 0 \le t \le 1, \\ 3, & 1 \le t < 2, \\ 4t^2, & 2 \le t \le 4, \end{cases}$$

where t is measured in hours from the clinic's opening time, what is the probability that 5 customers arrive in the first 2 hours?

(c) Shocks occur to a system according to a Poisson process of intensity $\lambda = 2$. Each shock causes some damage to the system and these damages accumulate. Let $N(t)$ be the number of shocks up to time t, and let Z_i be the damage caused by the i^{th} shock. Then

$$X(t) = Z_1 + \cdots + Z_{N(t)}$$

is the total damage up to time t. Determine the mean and variance of the total damage at time t when the individual shock damages are exponentially distributed with parameter α.

Question 5: 10[3+3+4]

(a) Let $(B(t))_{t\ge0}$ be a standard Brownian Motion. Find the number c for which $Pr\{B(4) > c|B(0) = 1\}$.

(b) Show that, for a standard Brownian Motion $(B(t))_{t\ge0}$ with $B(0) = 0$, and for any $s, t > 0$, we have $\text{Cov}[B(t), B(s)] = \min(t, s)$.

(c) Let $(B(t))_{t\ge0}$ be a standard Brownian Motion, show that the process defined by $W(t) = tB\left(\frac{1}{t}\right)$, with $W(0) = 0$, is a standard Brownian Motion too.

NORMAL DISTRIBUTION TABLE

Entries represent the area under the standardized normal distribution from -∞ to z, Pr(Z<z)

The value of z to the first decimal is given in the left column. The second decimal place is given in the top row.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Values of z for selected values of Pr(Z<z)							
z	0.842	1.036	1.282	1.645	1.960	2.326	2.576
Pr(Z<z)	0.800	0.850	0.900	0.950	0.975	0.990	0.995