Q1

(a) (i) $P_r\{X_2=1, X_3=1 \mid X_1=0\}$

$= P_r\{X_3=1 \mid X_2=1, X_1=0\} \cdot P_r\{X_2=1 \mid X_1=0\}$ Conditional property

↓ Markov property

$= P_r\{X_3=1 \mid X_2=1\} \cdot P_r\{X_2=1 \mid X_1=0\}$

$= P_{11} \cdot P_{01}$

$= (0.6)\ (0.2) = 0.12$

(ii) $P_r\{X_3=1 \mid X_0=0\}$

$= P_{01}^{(3)} = P_{0\,row} \cdot P^{(2)}_{1\,column}$

$= \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix} \begin{bmatrix} 0.26 \\ 0.36 \\ 0.1 \end{bmatrix} = (0.7 \times 0.26) + (0.2 \times 0.36) + (0.1 \times 0.1)$

$= 0.264$

(b) $X_0 = 1$, $X_n = p^{-n} \varepsilon_1 \cdots \varepsilon_n$, $\varepsilon_i \sim$ Bernolli(p)

$\Rightarrow E[\varepsilon_i] = p$

(1) $E|X_n| = E|p^{-n} \varepsilon_1 \cdots \varepsilon_n|$

$= E[p^{-n} \varepsilon_1 \cdots \varepsilon_n]$

$= p^{-n} E[\varepsilon_1 \cdots \varepsilon_n]$

$= p^{-n} E[\varepsilon_1] \cdots E[\varepsilon_n]$ (ε_i's are independent)

$= p^{-n} \underbrace{p \cdots p}_{n\text{-times}} = p^{-n} \cdot p^{n} = p^{0} = 1 < \infty$ ✓

(2) $E[X_{n+1} \mid F_n] = E[p^{-(n+1)} \varepsilon_1 \cdots \varepsilon_n \cdot \varepsilon_{n+1} \mid F_n]$

$= E[(p^{-n} \varepsilon_1 \cdots \varepsilon_n) \cdot p^{-1} \varepsilon_{n+1} \mid F_n]$

is F_n-adapted

$= p^{-n} \varepsilon_1 \cdots \varepsilon_n E[p^{-1} \varepsilon_{n+1} \mid F_n]$

$= p^{-n} \varepsilon_1 \cdots \varepsilon_n\, p^{-1} \underbrace{E[\varepsilon_{n+1}]}_{p}$ as ε_{n+1} indep. of F_n

$= p^{-n} \varepsilon_1 \cdots \varepsilon_n = X_n$ ✓

$\Rightarrow$ (1) & (2) implies martingale

Q2

(a) (i)

$$P = \begin{array}{c|cccc} & (S,S) & (S,C) & (C,S) & (C,C) \\ \hline (S,S) & 0.7 & 0.3 & 0 & 0 \\ (S,C) & 0 & 0 & 0.4 & 0.6 \\ (C,S) & 0.5 & 0.5 & 0 & 0 \\ (C,C) & 0 & 0 & 0.2 & 0.8 \\ \hline & \downarrow & \downarrow & \downarrow & \downarrow \\ & \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{array}$$

(ii) limiting distributi is $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$\begin{cases} 0.7\,\pi_0 + 0.5\,\pi_2 = \pi_0 \Rightarrow \pi_2 = 0.6\,\pi_0 \\ 0.3\,\pi_0 + 0.5\,\pi_2 = \pi_1 \Rightarrow \pi_1 = 0.6\,\pi_0 \\ 0.6\,\pi_1 + 0.8\,\pi_3 = \pi_3 \Rightarrow \pi_3 = 1.8\,\pi_0 \\ \& \ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

$\Longrightarrow \pi_0 = \frac{1}{4} = 0.25$

$\Longrightarrow \pi = (0.25, 0.15, 0.15, 0.45]$

$\therefore$ The long run fraction of days in which it is cloudy is $\pi_2 + \pi_3 = 0.15 + 0.45 = 0.60$

& The long run fraction of days in which it is Sunny is $\pi_0 + \pi_1 = 0.25 + 0.15 = 0.40$

Q2 <u>(b)</u>

$$\vartheta_{03} = E[T \mid X=0] \,?$$

$$\vartheta_{03} = 1 + P_{00}\vartheta_{03} + P_{01}\vartheta_{13} + P_{02}\vartheta_{23}$$

$$\Rightarrow \vartheta_{03} = 1 + 0.3\,\vartheta_{03} + 0.4\,\vartheta_{13} + 0.2\,\vartheta_{23} \longrightarrow (1)$$

$$\vartheta_{13} = 1 + \cancel{P_{10}}^{0}\vartheta_{03} + P_{11}\vartheta_{13} + P_{12}\vartheta_{23}$$

$$\Rightarrow \vartheta_{13} = 1 + 0.7\,\vartheta_{13} + 0.1\,\vartheta_{23} \longrightarrow (2)$$

$$\vartheta_{23} = 1 + \cancel{P_{20}}^{0}\vartheta_{03} + \cancel{P_{21}}^{0}\vartheta_{13} + P_{22}\vartheta_{23}$$

$$\Rightarrow \vartheta_{23} = 1 + 0.9\,\vartheta_{23} \longrightarrow (3)$$

Solve (1), (2) & (3) and find that:

$$\vartheta_{03} = \frac{170}{21} \approx \underline{\underline{8}}$$

$$0.7\,\vartheta_{03} = 1 + 0.4\,\vartheta_{13} + 0.2\,\vartheta_{23}$$

$$0.3\,\vartheta_{13} = 1 + 0.1\,\vartheta_{23}$$

$$0.1\,\vartheta_{23} = 1 \implies \vartheta_{23} = \frac{1}{0.1} = 10$$

$$\implies 0.3\,\vartheta_{13} = 1 + 1 = 2$$

$$\implies 3\,\vartheta_{13} = 20 \implies \vartheta_{13} = \frac{20}{3}$$

$$\implies 7\,\vartheta_{03} = 10 + 4\,\vartheta_{13} + 2\,\vartheta_{23}$$

$$\implies 7\,\vartheta_{03} = 10 + 4 \cdot \frac{20}{3} + 2 \cdot 10$$

$$\implies 7\,\vartheta_{03} = 10 + \frac{80}{3} + 20 = 30 + \frac{80}{3} = \frac{90+80}{3}$$

$$\implies \vartheta_{03} = \frac{170}{7 \times 3} = \frac{170}{21} \approx \underline{\underline{8}}$$

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Q3

(a) Now, $X(0) = 0 \Rightarrow P_0(0) = 1$

$$\Rightarrow P_n(0) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

we solve (1): $\dfrac{dP_0(t)}{dt} = -\lambda P_0(t)$

$$\Rightarrow \frac{dP_0(t)}{P_0(t)} = -\lambda dt$$

$$\Rightarrow \int_0^t \frac{dP_0(u)}{P_0(u)} = -\lambda \int_0^t du$$

$$\Rightarrow \ln P_0(u) \Big|_0^t = -\lambda u \Big|_0^t$$

$$\Rightarrow \ln P_0(t) - \ln \underbrace{P_0(0)}_{=1} = -\lambda t$$

$$\Rightarrow \ln P_0(t) = -\lambda t \Rightarrow \boxed{P_0(t) = e^{-\lambda t}}$$

solve (2): $\dfrac{dP_n(t)}{dt} + \lambda P_n(t) = \lambda P_{n-1}(t)$

multiply by $e^{\lambda t}$ both sides

$$\underbrace{e^{\lambda t}\left[\frac{dP_n(t)}{dt} + \lambda P_n(t)\right]}_{} = \lambda P_{n-1}(t) e^{\lambda t}$$

$$\parallel$$

$$\frac{d}{dt}\left[e^{\lambda t} P_n(t)\right] = \lambda P_{n-1}(t) e^{\lambda t}$$

$$\Rightarrow \int_0^t d\left[e^{\lambda x} P_n(x)\right] = \lambda \int_0^t P_{n-1}(x) e^{\lambda x} dx$$

$$\Rightarrow e^{\lambda x} P_n(x) \Big|_0^t = \lambda \int_0^t P_{n-1}(x) e^{\lambda x} dx$$

$$\Rightarrow e^{\lambda t} P_n(t) - \underbrace{P_n(0)}_{=0,\, n \geq 1} = \lambda \int_0^t P_{n-1}(x) e^{\lambda x} dx \quad , n = 1, 2, \ldots$$

$$\Rightarrow P_n(t) = \lambda e^{-\lambda t} \int_0^t P_{n-1}(x) e^{\lambda x} dx \quad , n = 1, 2, 3, \ldots$$

the recurrence relalation!

at $n=1 \Rightarrow P_1(t) = \lambda e^{-\lambda t} \int_0^t P_0(x) e^{\lambda x} dx$

$= \lambda e^{-\lambda t} \int_0^t e^{-\lambda x} e^{\lambda x} dx = \lambda e^{-\lambda t} \int_0^t dx$

$\Rightarrow \boxed{P_1(t) = \frac{(\lambda t)^1 e^{-\lambda t}}{1!}}$

$n=2 \Rightarrow P_2(t) = \lambda e^{-\lambda t} \int_0^t \lambda x e^{-\lambda x} e^{\lambda x} dx$

$= \lambda^2 e^{-\lambda t} \int_0^t x \, dx$

$= \lambda^2 e^{-\lambda t} \left. \frac{x^2}{2} \right|_0^t$

$\Rightarrow \boxed{P_2(t) = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}}$

- - - - and so on , so that $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$,

$n = 0, 1, 2, \ldots$

Q3 (b)

$X(0) = 3 \qquad \mu_0 = 0, \; \mu_1 = 2, \; \mu_2 = 3, \; \mu_3 = 5$

$P_n(t) \qquad n = 1, 2, 3$

$$\boxed{P_3(t) = e^{-\mu_3 t} = e^{-5t}}$$

$$P_2(t) = \mu_3 \left[A_{2,2} e^{-\mu_2 t} + A_{3,2} e^{-\mu_3 t} \right]$$

$$A_{3,2} = \prod_{i=2}^{3} \frac{1}{\mu_i - \mu_3} \qquad i \neq 3$$

$$= \frac{1}{\mu_2 - \mu_3} = \frac{1}{3-5} = -\frac{1}{2}$$

$$A_{2,2} = \frac{1}{\mu_3 - \mu_2} = \frac{1}{5-3} = \frac{1}{2}$$

$$\Rightarrow P_2(t) = 5 \left[\frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t} \right]$$

$$\boxed{P_2(t) = \frac{5}{2} e^{-3t} - \frac{5}{2} e^{-5t}}$$

$$P_1(t) = \mu_2 \mu_3 \left[A_{1,1} e^{-\mu_1 t} + A_{2,1} e^{-\mu_2 t} + A_{3,1} e^{-\mu_3 t} \right]$$

$$A_{1,1} = \prod_{i=1}^{3} \frac{1}{\mu_i - \mu_1} \qquad i \neq 1$$

$$\Rightarrow \boxed{P_1(t) = 15 \left[\frac{1}{3} e^{-2t} - \frac{1}{3} e^{-3t} + \frac{1}{6} e^{-5t} \right]} \checkmark$$

$$= \frac{1}{\mu_2 - \mu_1} \cdot \frac{1}{\mu_3 - \mu_1} = \frac{1}{3-2} \cdot \frac{1}{5-2} = \frac{1}{3}$$

$$A_{2,1} = \frac{1}{\mu_3 - \mu_2} \cdot \frac{1}{\mu_1 - \mu_2} = \frac{1}{5-2} \cdot \frac{1}{2-3}$$

$$= -\frac{1}{3}$$

$$A_{3,1} = \frac{1}{\mu_1 - \mu_3} \cdot \frac{1}{\mu_2 - \mu_3} = \frac{1}{2-5} \cdot \frac{1}{3-5} = -\frac{1}{3} \cdot \frac{-1}{2}$$

$$= \frac{1}{6}$$

(4) (a)

$$Pr\left(X(s+t) - X(s) = k\right) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

(i) $Pr\{X(2) = 3\}$

$$= Pr\{\underset{s+t}{X(2)} - \underset{s}{X(0)} = \underset{k}{3}\} \qquad \lambda = 3$$

$$= \frac{(3*2)^3 \; e^{-3*2}}{3!}$$

$$= \frac{6^3 \; e^{-6}}{3!} = 36 \; e^{-6}$$

(ii) $Pr\{X(2) = 3 \text{ and } X(4) = 7\}$

$$= Pr\{\underbrace{X(2) - X(0) = 3 \; , \; X(4) - X(2)}_{\text{indep.}} = 7 - 3 = 4\}$$

$$= Pr\{X(2) - X(0) = 3\} \cdot Pr\{X(4) - X(2) = 4\}$$

$$= 6^2 e^{-6} \quad * \quad \frac{6^4 \; e^{-6}}{4!}$$

$$= \frac{6^6 \; e^{-12}}{4!}$$

(iii) $Pr\{X(4) = 7 \mid X(2) = 3\}$

$$= Pr\{\underbrace{X(4) - X(2) = 4 \mid X(2) - X(0)}_{\underline{\text{indep.!}}} = 3\}$$

$$= Pr\{X(4) - X(2) = 4\}$$

$$= \frac{6^4 \; e^{-6}}{4!}$$

Q4 (b) $\Pr\{X(2)=5\}$

$$= \Pr\{X(2)-X(0)=5\} = \frac{\mu^k e^{-\mu}}{k!} = \frac{5^5 e^{-5}}{5!}$$

$$\mu = \int_0^2 \lambda(u)\,du$$

$$= \int_0^1 (2t+1)\,dt + \int_1^2 3\,dt$$

$$= 2 + 3 = 5$$

Q4 (c) $\lambda = 2$ $\quad \mu = E[Z_i]$, $\nu^2 = Var[Z_i]$

Rule of Compound Poisson Process

$$E[X(t)] = \lambda \mu t$$

$$Var[X(t)] = \lambda(\nu^2 + \mu^2)t$$

Now, each damage $Z_i \sim \exp(\alpha) \Rightarrow E[Z_i] = \mu = \frac{1}{\alpha}$
and $Var[Z_i] = \nu^2 = \frac{1}{\alpha^2}$

So $E[X(t)] = \lambda \mu t = 2\left(\frac{1}{\alpha}\right)t = \frac{2t}{\alpha}$

and $Var[X(t)] = \lambda(\nu^2+\mu^2)t$

$$= 2\left(\frac{1}{\alpha^2} + \frac{1}{\alpha^2}\right)t$$

$$= \frac{4t}{\alpha^2}$$

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Q5

(a) $P_r\{B(4) > c \mid B(0) = 1\} = 0.1587$

$$1 - P_r\{\underset{s+t}{B(4)} \le \underset{y}{c} \mid \underset{s}{B(0)} = \underset{x}{1}\} = 0.1587$$

$$1 - \Phi\left(\frac{c-1}{\sqrt{4}}\right) = 0.1587$$

$$\Rightarrow \Phi\left(\frac{c-1}{2}\right) = 1 - 0.1587$$

$$\Rightarrow \Phi\left(\frac{c-1}{2}\right) = 0.8413$$

$$\Rightarrow \frac{c-1}{2} = 1$$

$$\Rightarrow c - 1 = 2$$

$$\Rightarrow c = 2+1 = 3$$

$$c = 3$$

Q 5

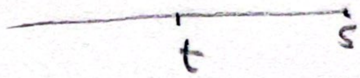
(b) $\operatorname{Cov}[B(t), B(s)] = \min(t, s)$

Recall $B(t) \sim N(0, t)$

$$\operatorname{Cov}[B(t), B(s)] = E[B(s)B(t)] - \underbrace{E[B(s)]}_{0}\underbrace{E[B(t)]}_{0}$$

$$= E[B(s)B(t)]$$

case ① $t < s$



$$B(s) = B(t) + B(s) - B(t)$$

$$E[B(s)B(t)] = E[B(t)(B(t) + B(s) - B(t))]$$

$$= E[B^2(t) + B(t)\cdot(B(s) - B(t))]$$

$$= E[B^2(t) + B(t)\cdot(B(s) - B(t))]$$

indep.

as $\operatorname{var}[B(t)] = t$

$$= t + \underbrace{E[B(t)]}_{0}E[B(s) - B(t)]$$

$$= t = \min(t, s)$$

case ② $s < t$ similarly.

and get that $\operatorname{Cov}[B(t), B(s)] = s = \min(t, s)$

Q5) $W(t) = t B(\frac{1}{t})$, $W(0)=0$ is a BM

1. $W(t) - W(s) = t B(\frac{1}{t}) - s B(\frac{1}{s})$

$$\sim N(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$$

$$= N(0, \underbrace{t + s - 2\sqrt{\tfrac{s}{t}}\sqrt{t}\sqrt{s}}_{= t - s})$$

$$\sim N(0, t-s)$$

explain steps !!

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2. $\text{cov}[W(t), W(s)] = \text{cov}[t B(\frac{1}{t}), s B(\frac{1}{s})]$

$= t \cdot s \, \text{cov}[B(\frac{1}{t}), B(\frac{1}{s})]$

$= \cancel{t} \cdot s \cdot \frac{1}{\cancel{t}} = s = \min(s,t)$

Case 1 $\longrightarrow$
$s \le t \Rightarrow \frac{1}{s} \ge \frac{1}{t}$

Case 2
Similarly for when $t \le s \Rightarrow \frac{1}{t} \ge \frac{1}{s}$.
we can prove: $\text{Cov}[W(t), W(s)] = \min(s,t)$

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3. $E[W(t)] = E[t B(\frac{1}{t})] = t E[B(\frac{1}{t})]$
$= 0$

as $B(\frac{1}{t}) \sim N(0, \frac{1}{t})$

1, 2 & 3 proves $W(t)$ is a BM.