King Saud University
College of Sciences
Department of Mathematics

Second Mid Term Exam, S1 1445
Math 380 – Stochastic Processes
Time: 2 hours

Answer 5 questions: 1, 2 and 3 are mandatory, then choose 2 of the other questions.

Q1: [1+4]

(a) Define a martingale.

(b) If $Y_1, Y_2, \ldots, Y_n, ..$ are independent random variables with $E|Y_i| < \infty$ for all i, and with constants $a_i = E[Y_i] \neq 0$ for all i, then prove that

$$Z_n = \frac{Y_1 Y_2 \ldots . Y_n}{a_1\, a_2 \ldots . a_n}, \quad n = 1,2, \ldots .$$

defines a martingale.

Q2: [2+2]

(a) The probability of the thrower winning in the dice game is p=0.5071. Suppose player A is the thrower and begins the game with \$5, and player B, his opponent, begins with \$10. What is the probability that player A goes bankrupt before player B? Assume that the bet is \$1 per round.

(b) What if p=0.5, does the probability that player A goes bankrupt before player B change or stays the same. Justify your answer.

Q3: [a(2+1.5+1) + b(1.5)]

(a) Consider the Markov chain whose transition probability matrix is given by

$$
\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}
$$

(i) Sketch, the Markov chain diagram, and determine whether it is an absorbing chain or not.

(ii) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(iii) Determine the mean time to absorption.

(b) Let X_n denote the quality of the nth item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that X_n evolves as a Markov chain whose transition matrix is

$$
\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{array}
$$

What is the probability that the third item is defective given that the first item is defective?

Q4: [2+3]

(a) For the Markov process $\{X_t\}$, t=0,1,2,...,n with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$. Prove that:

$$\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}, \text{ where } p_{i_0} = Pr\{X_0 = i_0\}.$$

(b) A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$
\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}
$$

and initial distribution $p_0=0.3$, $p_1=0.5$ and $p_2=0.2$.

Determine the probabilities $\Pr\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $\Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$.

Q5: [5]

A Markov chain $X_0, X_1, X_2,$ .... has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.5 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{Vmatrix} \end{array}$$

Every period that the process spends in state 0 incurs a cost \$4. Every period that the process spends in state 1 incurs a cost of \$7. Every period that the process spends in state 2 incurs a cost of \$5. What is the long run mean cost per period associated with this Markov chain?

Q6: [2 + 3]

(a) A Markov chain $X_0, X_1, X_2, \ldots$ with states 0, 1, and 2, has the transition probability matrix

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.3 \end{Vmatrix} \end{array}$$

and initial distribution $p_0 = Pr\{X_0 = 0\} = 0.4$ and $p_1 = Pr\{X_0 = 1\} = 0.6$. Determine $Pr\{X_2 = 0\}$.

(b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with

$Pr\{\xi_n = 0\} = 0.1, \quad Pr\{\xi_n = 1\} = 0.4, Pr\{\xi_n = 2\} = 0.3, Pr\{\xi_n = 3\} = 0.2,$

and suppose that $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end-of-period n .