King Saud University
College of Science
Department of Mathematics

Second Mid-Term Exam, S1, 1446 H.
Math-380 – Stochastic Processes
Time: 2 hours

Answer all questions from 1 to 5 , Question 6 is worth 2 bonus marks:

Question 1: [2+2+1]

A Markov chain $X_0, X_1, X_2, \ . \ . \ . \ . \ .$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and the following initial distribution:

$$p_0 = Pr\{X_0 = 0\} = 0.3 \ , \ p_1 = Pr\{X_0 = 1\} = 0.5 \ , \text{ and } \ p_2 = Pr\{X_0 = 2\} = 0.2 \, .$$

(a) Determine the probability $Pr\{ X_1 = 1\}$.

(b) Determine the probabilities:

$Pr\{X_0 = 1, X_1 = 1, X_2 = 0\}$ and $Pr\{X_1 = 1, X_2 = 1, X_3 = 0\}$.

(c) If the process starts from $X_0 = 2$, then find $Pr\{X_0 = 2, X_1 = 0, X_2 = 1\}$.

Question 2: [2+3]

(a) Let X_n denote the quality of the n^{th} item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that X_n evolves as a Markov chain whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array} \right\| \end{array}$$

What is the probability that the third item is defective given that the first item is defective?

(b) Consider a spare parts inventory model in which either 0, 1, 2 or 3 repair parts are demanded in any period, with

$$Pr\{\xi_n = 0\} = 0.1 \;,\; Pr\{\xi_n = 1\} = 0.4 \;,\; Pr\{\xi_n = 2\} = 0.3 \; and \; Pr\{\xi_n = 3\} = 0.2 \,,$$

and suppose that $s = 0$ and $S = 3$. Determine the transition probability matrix of the Markov chain $\{X_n\}$, where X_n represents the quantity on hand at the end of period n.

Question 3: [5]

Suppose that a parent has no offspring with probability $\frac{1}{3}$ and has two offsprings with a probability $\frac{2}{3}$. If a population of such individuals begins with a single parent and evolves as a branching process, determine u_n, the probability that the population will be extinct by the n^{th} generation for $n = 1, 2, 3, 4, 5$.

Question 4: [5]

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Sketch the Markov chain diagram, and determine whether it is an absorbing chain or not.
(ii) Starting in state 2, determine the probability that the Markov chain ends in state 0.
(iii) Determine the mean time to absorption if the process starts in the state 2.

Question 5: [5]

A Markov chain $X_0, X_1, X_2, \ \ . \ . \ . \ . \ .$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{Vmatrix} 0.5 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.5 \end{Vmatrix} \end{array}$$

Suppose that every period that the process spends in state 0 incurs a cost \$5, and every period that the process spends in state 1 incurs a cost of \$8, while every period that the process spends in state 2 incurs a cost of \$6. What is the long-run mean cost per period associated with this Markov chain?

Question 6: [Bonus 2 marks]

Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with a probability of 0.4, and is followed by a defective item with a probability of 0.6. Similarly, a defective item is followed by another defective item with a probability of 0.2, and is followed by a good item with a probability of 0.8. If the first item is good, what is the probability that the first defective item to appear is the fourth item?

Good Luck