King Saud University
College of Sciences
Department of Mathematics

Mid Term Exam, S3 1444
M 380 – Stochastic Processes
Time: 2 hours - Marks: 30

Answer the following questions.

Q1 [5]

Given the joint probability mass function of two random variables X and Y

as in the following table:

$X \backslash Y$	-1	0	1
-1	1/9	0	2/9
0	2/9	1/9	0
1	0	2/9	1/9

(i) Find $\rho(X,Y)$

(ii) Determine whether X and Y are two independent random variables or not, Justify your answer.

Q2: [4+4]

(a) Suppose that $\xi_1, \xi_2, \ldots$ are independent and identically distributed with $\Pr\{\xi_k = \pm 1\} = \frac{1}{2}$. Let N be independent of $\xi_1, \xi_2, \ldots$ and follow the geometric probability mass function $P_N(k) = \alpha(1-\alpha)^k$ for $k = 0, 1, \ldots,$ where $0 < \alpha < 1$. Form the random sum $Z = \xi_1 + \xi_2 + \ldots + \xi_N$. Determine the mean and variance of Z.

(b) Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \zeta_1 + \zeta_2 + \cdots + \zeta_n$ be the sum of n independent random variables, each exponentially distributed with mean $E(\zeta_k) = 1$. Show that $X_n = 2^n e^{-S_n}$, $n \geq 0$, defines a martingale.

Q3: [10:2+2+2+2+2]

(a) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P}=\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 1, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

(b) Let X_n denote the quality of the nth item that produced in a certain factory with $X_n = 0$ meaning "defective" and $X_n = 1$ meaning "good". Suppose that $\{X_n\}$ be a Markov chain whose transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 0.89 & 0.11 \\ 0.02 & 0.98 \end{array} \right\| \end{array}$$

(i) What is the probability that the fourth item is defective given that the first item is good?

(ii) In the long run, what is the probability that an item produced by this system is good or it's defective?

Q4: [7:2+2+3]

(a) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α . Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1-\alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(ii) Determine the probability that a correct signal is received at stage 2.

(b) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.3$, $\Pr\{\xi_n = 1\} = 0.2$, $\Pr\{\xi_n = 2\} = 0.5$ and suppose $s = 0$ and $S = 3$.

Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

The Model Answer

Q1: [5]

Y \ X	-1	0	1	$P_X(x)$
-1	1/9	0	2/9	1/3
0	2/9	1/9	0	1/3
1	0	2/9	1/9	1/3
$P_Y(y)$	1/3	1/3	1/3	Sum=1

$E(X)=0,\ E(X^2)=\frac{2}{3},\ Var(X)=\frac{2}{3}$

$E(Y)=0,\ E(Y^2)=\frac{2}{3},\ Var(Y)=\frac{2}{3}$

$E(XY)=0$

$Cov(X,Y)=E(XY)-E(X)E(Y)=0$

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$
$$=\frac{0}{2/3}$$
$$=0$$

$\Rightarrow X$ and Y are not correlated

$\because$ For example, $P(X=0,Y=1)=0,$ but $P(X=0)P(Y=1)\ =\frac{1}{9}$

$\Rightarrow P(X=0,Y=1)\neq P(X=0)P(Y=1)$

$\therefore\ X$ and Y are not independent r.vs

Q2: [4+4]

(a) We have,

$\boldsymbol{\xi}_k$	-1	1
$\Pr(\boldsymbol{\xi}_k)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\because \boldsymbol{\mu} = E(\boldsymbol{\xi}_k) = \sum_k \boldsymbol{\xi}_k \Pr(\boldsymbol{\xi}_k)$$

$$\therefore \boldsymbol{\mu} = -1(\tfrac{1}{2}) + 1(\tfrac{1}{2}) = 0$$

Also, $E(\boldsymbol{\xi}_k{}^2) = (-1)^2(\tfrac{1}{2}) + (1)^2(\tfrac{1}{2}) = 1$

and $\sigma^2 = Var(\xi_k) = E(\xi_k{}^2) - [E(\xi_k)]^2 = 1 - 0 = 1$

$$\because N \sim geom(\alpha),\ P_N(k) = \alpha(1-\alpha)^k,\ k = 0,1,\ldots$$

$$\therefore v = E(N) = \tfrac{1-\alpha}{\alpha},\ \tau^2 = Var(N) = \frac{1-\alpha}{\alpha^2}$$

$$\because Z = \xi_1 + \xi_2 + \ldots + \xi_N$$

$$\therefore E(Z) = \mu v = 0(\tfrac{1-\alpha}{\alpha}) = 0$$

and

$$Var(Z) = v\sigma^2 + \mu^2\tau^2$$

$$= \tfrac{1-\alpha}{\alpha}(1) + 0\left(\frac{1-\alpha}{\alpha^2}\right)$$

$$\therefore Var(Z) = \tfrac{1-\alpha}{\alpha}$$

(b)

(1) $E[|X_n|] = E[X_n] = E[2^n\, e^{-S_n}]$

$= 2^n\, E[e^{-\zeta_1} \ldots . e^{-\zeta_n}]$

$= 2^n\, E[e^{-\zeta_1}] \ldots . E[e^{-\zeta_n}]$, as $\zeta_{i's}$ are independent

$= 2^n\, \frac{1}{2} \ldots \ldots \frac{1}{2} = \frac{2^n}{2^n} = 1$, as

$E\left[e^{-\zeta_n}\right] = \int_0^\infty e^{-x}\ e^{-x}\, dx$

$= \int_0^\infty e^{-2x}\ dx = \frac{1}{2}$

So, $E[|X_n|] = 1 < \infty$.

(2) $E[X_{n+1}|X_0,\dots\dots,X_n] = E[2^{n+1}\, e^{-S_{n+1}}|X_0,\dots\dots,X_n],\ S_{n+1} = S_n + \zeta_{n+1}$

$$= E\left[2^n\, e^{-S_n}\ 2\, e^{-\zeta_{n+1}}\middle|X_0,\dots\dots,X_n\right]$$

$$= 2^n\, e^{-S_n}\, E\left[2\, e^{-\zeta_{n+1}}\middle|X_0,\dots\dots,X_n\right]$$

$$= 2^n\, e^{-S_n}\, 2\, E[e^{-\zeta_{n+1}}],$$

as ζ_{n+1} is independent of $X_{i's}$,

$$E[X_{n+1}|X_0,\dots\dots,X_n] = 2^n\, e^{-S_n}\, 2\,.\tfrac{1}{2}$$

$$= 2^n\, e^{-S_n}$$

$$= X_n.$$

We have proved from (1) and (2) that X_n is a martingale.

Q3: [10:2+2+2+2+2]

(a)

$$\mathbf{P}= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

$u_i = pr\{X_T = 0|X_0 = i\}$ for i=1,2,

and v_i =E[$T|X_0 = i$] for i=1,2.

(i)

$$u_1 = p_{10} + p_{11}u_1 + p_{12}u_2$$
$$u_2 = p_{20} + p_{21}u_1 + p_{22}u_2$$

$\Rightarrow$

$$u_1 = 0.2 + 0.4u_1 + 0.3u_2$$
$$u_2 = 0.1 + 0.5u_1 + 0.3u_2$$

$\Rightarrow$

$$6u_1 - 3u_2 = 2 \qquad (1)$$
$$5u_1 - 7u_2 = -1 \qquad (2)$$

Solving (1) and (2), we get

$$u_1 = \tfrac{17}{27} \text{ and } u_2 = \tfrac{16}{27}$$

Starting in state 1, the probability that the Markov chain ends in state 0 is

$$u_1 = u_{10} = \tfrac{17}{27} = 0.6296$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.3v_2$$
$$v_2 = 1 + 0.5v_1 + 0.3v_2$$

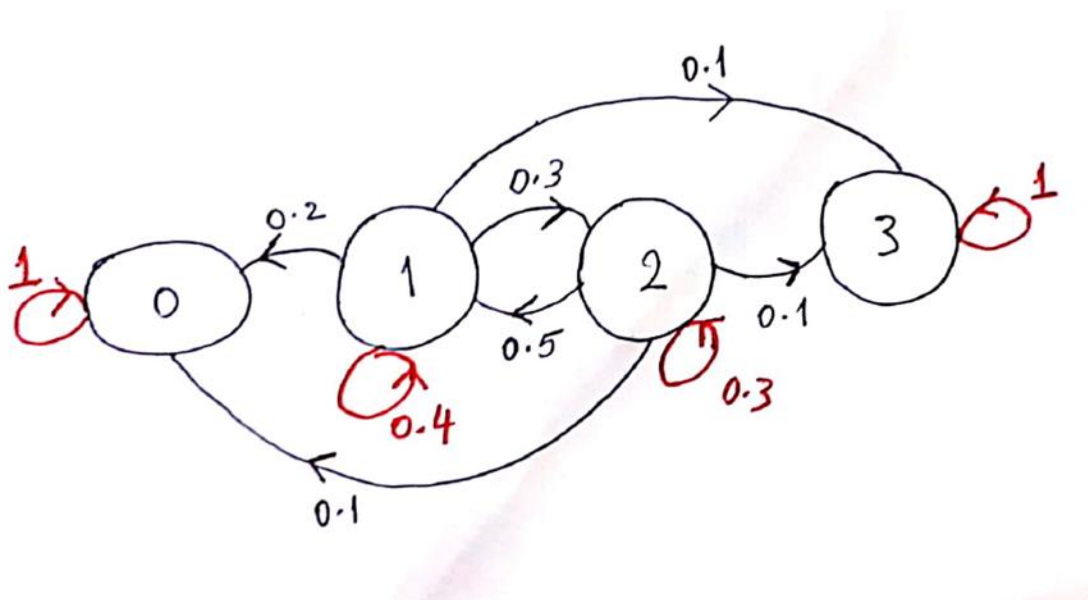
$\Rightarrow$

$$6v_1 - 3v_2 = 10 \qquad (1)$$
$$5v_1 - 7v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

$$v_1 = v_{10} = \tfrac{100}{27}$$
$$\approx 3.7037$$

(iii)



It's an absorbing Markov Chain.

(b)

(i)

$$P^2 = \begin{bmatrix} 0.89 & 0.11 \\ 0.02 & 0.98 \end{bmatrix} \begin{bmatrix} 0.89 & 0.11 \\ 0.02 & 0.98 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.7943 & 0.2057 \\ 0.0374 & 0.9626 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.7110 & 0.2890 \\ 0.0525 & 0.9475 \end{bmatrix}$$

$$pr\{X_3 = 0 | X_0 = 1\} = p_{10}^3 = 0.0525$$

$$= 5.25\% ,$$

which is the probability that the fourth item is defective given that the first item is good.

(ii) In the long run, the probability that an item produced by this system is defective is given by:

$$b/(a+b) = \frac{0.02}{0.02+0.11}$$

$$= \frac{2}{13} = 15.38\ \%$$

In the long run, the probability that an item produced by this system is good is given by:

$$a/(a+b) = \frac{0.11}{0.02+0.11}$$
$$= \frac{11}{13} = 84.62\ \%\ ,$$

where $\lim_{n\to\infty} \mathrm{P}^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$, $\mathrm{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$

Clearly, $1-\frac{2}{13} = \frac{11}{13} = 84.62\ \%$

Q4: [7:2+2+3]

(a)

The transition probability matrix can be written as

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\| \begin{array}{cc} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{array} \right\| \end{array}$$

(i) The probability that no error occurs up to stage $n=2$ is given as follows.

$$\Pr\{X_0=0, X_1=0, X_2=0\} = p_0 P_{00} P_{00},\ \ \mathrm{p}_0 = pr\{X_0=0\} = 1$$
$$= 1\times(1\text{-}\alpha)\times(1\text{-}\alpha)$$
$$= (1\text{-}\alpha)^2$$

(ii) The probability that a correct signal is received at stage 2 is given as follows.

$$\Pr\{X_0=0, X_1=0, X_2=0\} + \Pr\{X_0=0, X_1=1, X_2=0\}$$
$$= p_0 P_{00} P_{00} + p_0 P_{01} P_{10}$$
$$= (1-\alpha)^2 + \alpha^2$$
$$= 1-2\alpha+2\alpha^2$$

(b)

$$\begin{array}{c} \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccccc} -1 & 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{ccccc} 0 & 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.5 & 0.2 & 0.3 \\ 0.5 & 0.2 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0.5 & 0.2 & 0.3 \end{array}\right\| \end{array}$$

P_{ij} =Pr$(\xi_{n+1} = S - j)$, $i \le s$ for replenishment

$P_{-1,-1}$=Pr$(\xi_{n+1} = 4) = 0$, P_{01}=Pr$(\xi_{n+1} = 2) = 0.5$

P_{ij} =Pr$(\xi_{n+1} = i - j)$, $s < i \le S$ for non-replenishment

$P_{1,-1}$=Pr$(\xi_{n+1} = 2) = 0.5$, P_{11}=Pr$(\xi_{n+1} = 0) = 0.3$, P_{21}=Pr$(\xi_{n+1} = 1) = 0.2$