King Saud University
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Department of Mathematics

Mid Term Exam, S1 1444
M 380 – Stochastic Processes
Time: 90 minutes - Marks: 30

Answer the following questions.

Q1: [5+5]

(a) Suppose that $\xi_1, \xi_2, \ldots$ are independent and identically distributed with $\Pr\{\xi_k = \pm 1\} = \frac{1}{2}$. Let N be independent of $\xi_1, \xi_2, \ldots$ and follow the geometric probability mass function

$P_N(k) = \alpha(1-\alpha)^k$ for $k = 0, 1, \ldots,$ where $0 < \alpha < 1$. Form the random sum $Z = \xi_1 + \xi_2 + \ldots + \xi_N$. Determine the mean and variance of Z.

(b) Let $S_0 = 0$, and for $n \geq 1$, let $S_n = \zeta_1 + \zeta_2 + \cdots + \zeta_n$ be the sum of n independent random variables, each exponentially distributed with mean $E(\zeta_k) = 1$. Show that: $X_n = 2^n e^{-S_n}$, $n \geq 0$, defines a martingale.

Q2: [5+5]

A Markov chain $X_0,\ X_1,\ X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}$$

(a) Determine the probability $pr\{X_0 = 1, X_1 = 0, X_2 = 2\}$, if it is known that the process starts in state $X_0 = 1$.

(b) Determine the conditional probability $\Pr\{X_2 = 1,\ X_3 = 1 | X_1 = 0\}$.

Q3: [5+5]

(a) A Markov chain $X_0,\ X_1,\ X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\Vert \begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array} \right\Vert \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probability $pr\{X_3 = 0\}$.

(b) Consider a spare parts inventory model in which either $0, 1,$ or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.5,\ \Pr\{\xi_n = 1\} = 0.4,\ \Pr\{\xi_n = 2\} = 0.1.$ and suppose $\mathrm{s} = 0$ and $S = 3.$ Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

The Model Answer

Q1: [5+5]

(a) We have,

ξ_k	-1	1
$\Pr(\xi_k)$	$\frac{1}{2}$	$\frac{1}{2}$

$\because \boldsymbol{\mu} = E(\xi_k) = \sum_k \xi_k \Pr(\xi_k)$

$\therefore \boldsymbol{\mu} = -1(\frac{1}{2}) + 1(\frac{1}{2}) = 0$

Also, $E(\xi_k{}^2) = (-1)^2(\frac{1}{2}) + (1)^2(\frac{1}{2}) = 1$

and $\sigma^2 = Var(\xi_k) = E(\xi_k{}^2) - [E(\xi_k)]^2 = 1 - 0 = 1$

$\because N \sim geom(\alpha),\ P_N(k) = \alpha(1-\alpha)^k,\ k = 0,1,\ldots$

$\therefore v = E(N) = \frac{1-\alpha}{\alpha},\ \tau^2 = Var(N) = \frac{1-\alpha}{\alpha^2}$

$\because Z = \xi_1 + \xi_2 + \ldots + \xi_N$

$\therefore E(Z) = \mu v = 0(\frac{1-\alpha}{\alpha}) = 0$

and

$$Var(Z) = v\sigma^2 + \mu^2\tau^2$$

$$= \frac{1-\alpha}{\alpha}(1) + 0\left(\frac{1-\alpha}{\alpha^2}\right)$$

$\therefore Var(Z) = \frac{1-\alpha}{\alpha}$

(b)

(1) $E[|X_n|] = E[X_n] = E[2^n\, e^{-S_n}]$

$= 2^n\, E[e^{-\zeta_1} \ldots e^{-\zeta_n}]$

$= 2^n\, E[e^{-\zeta_1}] \ldots E[e^{-\zeta_n}]$, as $\zeta_{i's}$ are independent

$= 2^n \frac{1}{2} \cdots\cdots \frac{1}{2} = \frac{2^n}{2^n} = 1$, as

$E[e^{-\zeta_n}] = \int_0^\infty e^{-x} \ e^{-x} \, dx$

$= \int_0^\infty e^{-2x} \ dx = \frac{1}{2}$

So, $E[|X_n|] = 1 < \infty$.

(2) $E[X_{n+1}|X_0, \ldots\ldots, X_n] = E[2^{n+1} e^{-S_{n+1}}|X_0, \ldots\ldots, X_n]$, $S_{n+1} = S_n + \zeta_{n+1}$

$= E[2^n e^{-S_n} \ 2 e^{-\zeta_{n+1}}|X_0, \ldots\ldots, X_n]$

$= 2^n e^{-S_n} E[2 e^{-\zeta_{n+1}}|X_0, \ldots\ldots, X_n]$

$= 2^n e^{-S_n} 2 E[e^{-\zeta_{n+1}}]$,

as ζ_{n+1} is independent of $X_{i's}$,

$E[X_{n+1}|X_0, \ldots\ldots, X_n] = 2^n e^{-S_n} 2 . \frac{1}{2}$

$= 2^n e^{-S_n}$

$= X_n$.

We have proved from (1) and (2) that X_n is a martingale.

Q2: [5+5]

(a)

$\because pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = p_1 P_{10} P_{02}$, $p_1 = pr\{X_0 = 1\} = 1$

$= 1(0.3)(0.1)$

$\therefore pr\{X_0 = 1, X_1 = 0, X_2 = 2\} = 0.03$

(b)

$\Pr\{X_2 = 1, \ X_3 = 1 | X_1 = 0\}$

$= \Pr\{X_3 = 1 | X_2 = 1, X_1 = 0\} . \Pr\{X_2 = 1 | X_1 = 0\}$ Conditional Prob. Property

$= \Pr\{X_3 = 1 | X_2 = 1\} . \Pr\{X_2 = 1 | X_1 = 0\} = P_{11} P_{01}$ Markov definition

$= 0.3(0.3)$

$= 0.09$

Q3: [5+4]

(a)

$$\because\ pr\{X_3=0\}=pr\{X_3=0|X_0=0\}\,pr\{X_0=0\}$$
$$+pr\{X_3=0|X_0=1\}\,pr\{X_0=1\}$$
$$=P_{00}^3P_0+P_{10}^3P_1\ ,\quad P_0=0.5,\ P_1=0.5$$

$$P^3=\begin{bmatrix}0.3 & 0.2 & 0.5\\ 0.5 & 0.1 & 0.4\\ 0.5 & 0.2 & 0.3\end{bmatrix}\begin{bmatrix}0.44 & 0.18 & 0.38\\ 0.40 & 0.19 & 0.41\\ 0.40 & 0.18 & 0.42\end{bmatrix}$$
$$=\begin{bmatrix}0.4120 & 0.1820 & 0.4060\\ 0.4200 & 0.1810 & 0.3990\\ 0.4200 & 0.1820 & 0.3980\end{bmatrix}$$
$$\therefore\ pr\{X_3=0\}=(0.412)(0.5)+(0.42)(0.5)$$
$$=0.416$$

(b)

The transition probability matrix is given by

$$\mathbf{P}=\begin{array}{c} \\ -1\\ 0\\ 1\\ 2\\ 3\end{array}\begin{array}{c}\begin{array}{ccccc}-1 & 0 & 1 & 2 & 3\end{array}\\ \left\|\begin{array}{ccccc}0 & 0 & 0.1 & 0.4 & 0.5\\ 0 & 0 & 0.1 & 0.4 & 0.5\\ 0.1 & 0.4 & 0.5 & 0 & 0\\ 0 & 0.1 & 0.4 & 0.5 & 0\\ 0 & 0 & 0.1 & 0.4 & 0.5\end{array}\right\|\end{array}$$

where,

$$P_{ij}=\Pr\{X_{n+1}=j|X_n=i\}$$
$$=\begin{cases}\Pr(\xi_{n+1}=3-j),\ i\le 0 & \text{replenishment}\\ \Pr(\xi_{n+1}=i-j),\ 0<i\le 3 & \text{without replenishment}\end{cases}$$
