King Saud University
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Department of Mathematics

Midterm Exam, S2 1444
M 380 – Stochastic Processes
Time: 2 hours - Marks: 30

Answer the following questions.

Q1: [3+3+3]

(a) The lifetime, in years, of a certain class of light bulbs has an exponential distribution with parameter $\lambda = 2$. What is the probability that a bulb selected at random from this class will last more than 1.5 years? What is the probability that a bulb selected at random will last exactly 1.5 years?

(b) Let $U_1, U_2, \ldots . U_n$ be independent random variables each uniformly distributed over the interval (0, 1]. Show that $X_0 = 1$ and $X_n = 2^n\, U_1\, U_2 \ldots . U_n$ for $n = 1,2,\ldots$ defines a martingale.

(c) Suppose that X and Y are two independent random variables with the geometric distribution

$p(k) = (1-\pi)\pi^k$ for $k = 0,1,\ldots$ find the probability mass function of $Z = X + Y$.

Q2: [3+4]

(a) For the Markov process $\{X_t\}$, $t = 0,1,2,\ldots,n$ with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \mathrm{pr}\{X_0 = i_0\}$

(b) If a Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5$, $p_1 = 0.2$ and $p_2 = 0.3$ Find $\mathrm{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$

Q3: [3+4]

(a) Let $\{X_n\}$ be a Markov chain with state space $S = \{0,1\}$ has the transition probability matrix

$\mathbf{P} = \left\| \begin{array}{cc} 0.5 & 0.5 \\ 1 & 0 \end{array} \right\|$, find $pr\{X_5 = 1 | X_2 = 0\}$.

(b) Consider a spare parts inventory model in which either $0, 1, 2$ repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.5$, $\Pr\{\xi_n = 1\} = 0.4$, $\Pr\{\xi_n = 2\} = 0.1$ and suppose $s = 0$ and $S = 3$. Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

Q4: [4+3]

(a) Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

(i) Starting in state 2, determine the probability that the Markov chain ends in state 0.

(ii) Determine the mean time to absorption.

(iii) Sketch, the Markov chain diagram, and determine whether it's an absorbing chain or not.

(b) Suppose that the social classes of successive generations in a family follow a Markov chain with transition probability matrix given by

		Son's class		
		Lower	Middle	Upper
Father's class	Lower	0.7	0.2	0.1
	Middle	0.2	0.6	0.2
	Upper	0.1	0.4	0.5

What fraction of families are upper class in the long run?

The Model Answer

Q1: [3+3+3]

(a)

$X \sim \exp(2)$

(i) $\Pr(T > 1.5) = e^{-3} = 0.0498$

(ii) $\Pr(T = 1.5) = 0$

(b)

1- As X_n is a non-negative random variable,

$$E[|X_n|] = E[X_n] = E[2^n\, U_1\, U_2 \ldots . U_n] = 2^n\, E[U_1] \ldots . E[U_n]$$
$$= 2^n . \left(\tfrac{1}{2}\right)^n = 1 < \infty.$$

This is because $U_i's$ are independent, also, since $U_i \sim uniform(0,1]$, then $E[U_i] = \frac{1}{2}$.

2- $E[X_{n+1}|X_0, \ldots, X_n] = E[2^{n+1}\, U_1\, U_2 \ldots . U_n . U_{n+1}|X_0, \ldots, X_n]$

$$= 2^n\, U_1\, U_2 \ldots . U_n\, E[2 . U_{n+1}|X_0, \ldots, X_n]$$

$$= 2^n\, U_1\, U_2 \ldots . U_n\;\; 2\, E[U_{n+1}|X_0, \ldots, X_n]$$

$$= 2^n\, U_1\, U_2 \ldots . U_n\;\; 2\, E[U_{n+1}]$$

$$= 2^n\, U_1\, U_2 \ldots . U_n\;\; 2 . \tfrac{1}{2} = 2^n\, U_1\, U_2 \ldots . U_n = X_n.$$

From 1 and 2, we have proved that X_n is a martingale.

(c)

$p(k) = (1-\pi)\pi^k$ for $k = 0,1,\ldots$

$$\Pr\{X + Y = n\} = \sum_{k=0}^{n} \Pr\{X = k, Y = n-k\} \quad \text{for } k = 0,1,\ldots$$

$$= \sum_{k=0}^{n} (1-\pi)\pi^k (1-\pi)\pi^{n-k},$$

where X and Y are 2-indpendent r.vs

$$\therefore \Pr\{X+Y=n\} = (1-\pi)^2\pi^n \sum_{k=0}^{n} 1$$
$$= (1-\pi)^2\pi^n (n+1)$$
$$= (n+1)(1-\pi)^2\pi^n, \ n \geq 0$$

Q2: [3+4]

(a)

$$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$$
$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$$
$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n} \quad \text{Definition of Markov}$$

By repeating this argument $n-1$ times

$$\therefore \ \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$$
$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

$$pr\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}, \quad p_1 = pr\{X_1 = 1\}$$
$$pr\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$$
$$= P_{01}p_0 + P_{11}p_1 + P_{21}p_2$$
$$= 0.3(0.5) + 0.2(0.2) + 0.3(0.3) = 0.28$$
$$\therefore \ pr\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.28(0.2)(0.4) = 0.0224$$

Q3: [3+4]

(a)

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$$
$$P^3 = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$
$$pr\{X_5 = 1 | X_2 = 0\} = p_{01}^3 = 3/8$$
$$= 0.375$$

(b)

$$
\mathbf{P} = \begin{array}{c|ccccc|}
 & -1 & 0 & 1 & 2 & 3 \\
-1 & 0 & 0 & 0.1 & 0.4 & 0.5 \\
0 & 0 & 0 & 0.1 & 0.4 & 0.5 \\
1 & 0.1 & 0.4 & 0.5 & 0 & 0 \\
2 & 0 & 0.1 & 0.4 & 0.5 & 0 \\
3 & 0 & 0 & 0.1 & 0.4 & 0.5
\end{array}
$$

where,

$$
P_{ij} = \Pr\left\{X_{n+1} = j \middle| X_n = i\right\}
= \begin{cases} \Pr(\xi_{n+1} = 3 - j), \ i \le 0 & \text{replenishment} \\ \Pr(\xi_{n+1} = i - j), \ 0 < i \le 3 & \text{without replenishment} \end{cases}
$$

Q4: [4+3]

(a)

$$
\mathbf{P}= \begin{array}{c|cccc|}
 & 0 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0.2 & 0.4 & 0.3 & 0.1 \\
2 & 0.1 & 0.5 & 0.3 & 0.1 \\
3 & 0 & 0 & 0 & 1
\end{array}
$$

$u_i = pr\left\{X_T = 0 \middle| X_0 = i\right\}$ for i=1,2,

and v_i =E[$T \middle| X_0 = i$] for i=1,2.

(i)

$$
u_1 = p_{10} + p_{11}u_1 + p_{12}u_2
$$
$$
u_2 = p_{20} + p_{21}u_1 + p_{22}u_2
$$

$\Rightarrow$

$$
u_1 = 0.2 + 0.4u_1 + 0.3u_2
$$
$$
u_2 = 0.1 + 0.5u_1 + 0.3u_2
$$

$\Rightarrow$

$$
6u_1 - 3u_2 = 2 \qquad (1)
$$
$$
5u_1 - 7u_2 = -1 \qquad (2)
$$

Solving (1) and (2), we get

$$u_1 = \frac{17}{27} \text{ and } u_2 = \frac{16}{27}$$

Starting in state 2, the probability that the Markov chain ends in state 0 is

$$u_2 = u_{20} = \frac{16}{27} = 0.5926$$

(ii) Also, the mean time to absorption can be found as follows

$$v_1 = 1 + p_{11}v_1 + p_{12}v_2$$
$$v_2 = 1 + p_{21}v_1 + p_{22}v_2$$

$\Rightarrow$

$$v_1 = 1 + 0.4v_1 + 0.3v_2$$
$$v_2 = 1 + 0.5v_1 + 0.3v_2$$

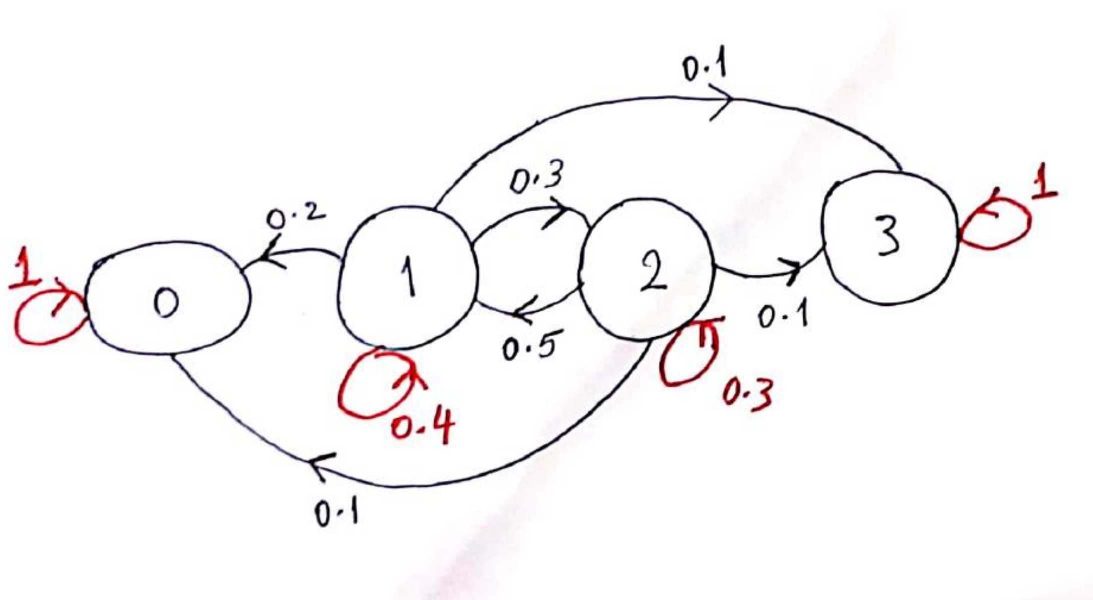
$\Rightarrow$

$$6v_1 - 3v_2 = 10 \qquad (1)$$
$$5v_1 - 7v_2 = -10 \qquad (2)$$

Solving (1) and (2), we get

$$v_2 = v_{20} = \frac{110}{27}$$
$$\approx 4.0741$$

(iii) It's an absorbing Markov Chain.



Markov Chain Diagram

(b)

Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the limiting distribution

For j=0, $\pi_0 = \pi_0 p_{00} + \pi_1 p_{10} + \pi_2 p_{20}$

Implies: $0.3\pi_0 - 0.2\pi_1 - 0.1\pi_2 = 0$

Which gives: $3\pi_0 - 2\pi_1 - \pi_2 = 0 \quad (1)$

Similarly, for j=1, $-2\pi_0 + 4\pi_1 - 4\pi_2 = 0 \quad (2)$

and by using: $\pi_0 + \pi_1 + \pi_2 = 1 \quad (3)$

Solving 1, 2 and 3 by Cramer's rule:

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -2 & 4 & -4 \\ 1 & 1 & 1 \end{vmatrix} = 34$$

Also,

$$\Delta_2 = \begin{vmatrix} 3 & -2 & 0 \\ -2 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 8$$

Therefore, fraction of families are upper class in the long run $= \pi_2 = \frac{\Delta_2}{\Delta} = \frac{4}{17}$.