King Saud University
College of Sciences
Department of Mathematics

Final Exam, S1 1443
M 380 - Stochastic Processes
Time: 3 hours - **Marks: 40**

Answer the following questions:

Q1: [3+6]

(a) Suppose that X is an exponentially distributed random variable with parameter λ.

(i) Prove that: $Pr(X > t + s | X > s) = Pr(X > t) \quad \forall t, s \geq 0$.

(ii) What is the name of this property?

(iii) If $X \sim exp(0.03)$, find $Pr(X \leq 81 | X > 70)$.

(b)

(i) For discrete bivariate random variables X and Y, what is the formula of conditional probability mass function of X given $Y=y$? Then, deduce the formula for marginal mass function of X.

(ii) Suppose X has a binomial distribution with parameters p and N, where N has a Poisson distribution with mean λ. What is the marginal distribution for X?

Q2: [3+3+2]

(a) An observation is made of a Poisson random variable N with parameter λ. Then N independent Bernoulli trials are performed, each with probability p of success. Let Z be the total number of successes observed in the N trials. Formulate Z as a random sum and determine its mean and variance. What is the distribution of Z?

(b) Let $\zeta_1, \zeta_2, \zeta_3, \ldots$.. be independent Bernoulli random variables with parameter p, $0<p<1$.

Show that $X_0 = 1$ and $X_n = p^{-n} \zeta_1 \zeta_2 \ldots \zeta_n$, $n = 1,2,\ldots$, defines a nonnegative martingale.

(c) Messages arrive at a telegraph office as a Poisson Process with mean rate of 3 messages per hour. What is the probability that no messages arrive during the morning hours 8:00 A.M. to noon?

Q3: [3+5]

(a) Consider a spare parts inventory model in which either 0, 1, or 2 repair parts are demanded in any period, with $\Pr\{\xi_n = 0\} = 0.3$, $\Pr\{\xi_n = 1\} = 0.2$, $\Pr\{\xi_n = 2\} = 0.5$ and suppose $s = 0$ and $S = 3$.

Determine the transition probability matrix for the Markov chain $\{X_n\}$, where X_n is defined to be the quantity on hand at the end of period n.

(b) Suppose that the weather on any day depends on the weather conditions for the previous 2 days. Suppose also that if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability 0.5; if it was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability 0.4; if it was sunny today and yesterday, then it will be sunny tomorrow with probability 0.7; if it was cloudy for the last 2 days, then it will be sunny tomorrow with probability 0.2. Transform this model into a Markov chain, and then find the transition probability matrix. Find also the long run fraction of days in which it is cloudy.

Q4: [4+4]

(a) Let X_n denote the condition of a machine at the end of period n for $n = 1, 2, \ldots$. Let X_0 be the condition of the machine at the start. Consider the condition of the machine at any time can be observed and classified as being in one of the following three states: **State** 0: Good operating order, **State** 1: Deteriorated operating order and **State** 2: In repair. Assume that $\{X_n\}$ is a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array}\right\| \end{array}$$

and starts in state $X_0 = 1$.

(i) Find $\Pr\{X_2 = 1\}$.

(ii) Calculate the limiting distribution.

(iii) What is the long run rate of repairs per unit time?

(b) A pure death process starting from $X(0) = 3$ has death parameters $\mu_0 = 0$, $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 5$. Determine $P_n(t)$ for $n = 0, 1, 2, 3$.

Q5: [5+2]

(a) If $X(t)$ represents a size of a population where $X(0) = 1$, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\,(p)$, $p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) Let $X(t)$ be a Yule process that is observed at a random time U, where U is uniformly distributed over [0,1). Show that $pr\{X(U)=k\} = p^k/(\beta k)$ for $k = 1, 2, \ldots$, with $p = 1-e^{-\beta}$.

Model Answer

Q1: [3+6]

(a)

(i) $Pr(X > t + s|X > s) = \frac{Pr(X>t+s, X>s)}{Pr(X>s)}$

$= \frac{Pr(X>t+s)}{Pr(X>s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}$

$= e^{-\lambda t} = Pr(X > t)$, where $X \sim exp(\lambda)$.

(ii) The memoryless property.

(iii) We use (i):

$Pr(X \leq 81|X > 70) = 1 - Pr(X > 81|X > 70)$

$= 1 - Pr(X > 11)$

$= Pr(X \leq 11)$

Therefore, $Pr(X \leq 81|X > 70) = 1 - e^{-0.03(11)}$

$= 1 - e^{-0.33}$

$\simeq 0.2811$

(b)

(i) $P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$ is the conditional probability mass function of X given Y=y, where $P_Y(y) \neq 0$. This implies that:

$P_{X,Y}(x,y) = P_{X|Y}(x|y)\ P_Y(y)$ (1)

Also, we have that

$P(X = x) = \sum_y P_{X,Y}(x,y)$ (2)

Combine (1) and (2) to get:

$P(X = x) = \sum_y P_{X|Y}(x|y)\ P_Y(y)$, which is the marginal mass function of X.

(ii)

Since, $X \sim Bin(p, N)$ and $N \sim Poisson(\lambda)$

So, $P_{X|N}(x|n) = \binom{n}{x} p^x\ (1-p)^{n-x}, x = 0,1,2,\dots,n$ and $P_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0,1,2,\dots$

So, $P(X = x) = \ \sum_n P_{X|N}(x|n)\ P_N(n) = \sum_n \binom{n}{x} p^x\ (1-p)^{n-x} \frac{e^{-\lambda}\lambda^n}{n!}$

$$= \lambda^x p^x\ e^{-\lambda} \sum_{n=x}^{\infty} \frac{(\lambda(1-p))^{n-x}}{x!\,(n-x)!},$$

Put $n - x = r$, to get

$$P(X = x)\ = \frac{(\lambda p)^x e^{-\lambda}}{x!} \sum_{r=0}^{\infty} \frac{(\lambda(1-p))^r}{r!},$$

Then by using $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$, we obtain

$$P(X = x) = \frac{(\lambda p)^x e^{-\lambda}}{x!}\ e^{\lambda(1-p)}$$

$$= \frac{(\lambda p)^x e^{-\lambda p}}{x!}\ ,\ x = 0,1,2,\dots$$

Therefore, $X \sim Poisson(\lambda p)$.

Q2: [3+3+2]

(a)

Let $Z = \zeta_1 + \zeta_2 + \dots + \zeta_N$,

$where\ \ N \sim Poisson(\lambda)$ and each $\zeta_k \sim Bernoulli(p)$.

Then, $(N) = \nu = \lambda, \quad Var(N) = \tau^2 = \lambda$,

$E(\zeta_k) = \mu = p\ and \quad Var(\zeta_k) = \sigma^2 = p(1-p)$.

By using $E(Z)\ =\ \mu\,\nu\,,\ Var(Z)\ = \nu\sigma^2\ + \mu^2\tau^2$,

we get that $E(Z) = \lambda p \quad and \quad Var(Z) = \lambda p$, that is $Z \sim Poisson(\lambda p)$.

(b)

A stochastic process X_n is a martingale if for $n = 0,1,2,\dots$

(1) $E[|X_n|] < \infty$,

(2) $E[X_{n+1}|X_0,\dots\dots,X_n]\ =\ X_n$.

$\Rightarrow$

(1) $E[|X_n|] = E[X_n] = E[p^{-n}\zeta_1\zeta_2 \dots \zeta_n]$, and as $\zeta_{i's}$ are independent,

$$= p^{-n}\, E[\zeta_1] \dots . E[\zeta_n]$$

$$= p^{-n}\, p^n = 1, \text{ as } E[X] = p,$$

$\therefore\ E[|X_n|] = E[X_n] = 1 < \infty.$

(2) $E[X_{n+1}|X_0, \dots\dots, X_n] = E[p^{-(n+1)}\zeta_1\zeta_2 \dots \zeta_{n+1}|X_0, \dots\dots, X_n]$

$$= E[p^{-n}\zeta_1\zeta_2 \dots \zeta_n\, p^{-1}\zeta_{n+1}|X_0, \dots\dots, X_n]$$

$$= p^{-n}\zeta_1\zeta_2 \dots \zeta_n\, E[p^{-1}\zeta_{n+1}|X_0, \dots\dots, X_n]$$

$$= p^{-n}\zeta_1\zeta_2 \dots \zeta_n\, p^{-1}E[\zeta_{n+1}|X_0, \dots\dots, X_n]$$

$E[X_{n+1}|X_0, \dots\dots, X_n] = p^{-n}\zeta_1\zeta_2 \dots \zeta_n\, p^{-1}E[\zeta_{n+1}]$, as ζ_{n+1} is independent of $X_{i's}$,

$$= p^{-n}\zeta_1\zeta_2 \dots \zeta_n\, p^{-1}\, p$$

$\therefore\ E[X_{n+1}|X_0, \dots\dots, X_n] = p^{-n}\zeta_1\zeta_2 \dots \zeta_n = X_n.$

We have proved from (1) and (2) that X_n defines a nonnegative martingale.

(c)

For Poisson Process $\Pr\{X(s+t) - X(s) = k\} = \dfrac{(\lambda t)^k e^{-\lambda t}}{k!},\ k = 0, 1, 2, \dots$

$$\therefore \Pr\{X(12) - X(8) = 0\} = \frac{(3\times 4)^0 e^{-3(4)}}{0!} = e^{-12}$$
$$\approx 6.1442\times 10^{-6}$$

Where $\lambda = 3,\ t = 12 - 8 = 4$ and $k = 0$

Q3: [3+5]

(a)

	-1	0	1	2	3
-1	0	0	0.5	0.2	0.3
0	0	0	0.5	0.2	0.3
1	0.5	0.2	0.3	0	0
2	0	0.5	0.2	0.3	0
3	0	0	0.5	0.2	0.3

P_{ij} =Pr($\xi_{n+1} = S - j$) , $i \le s$ for replenishment

$P_{-1,-1}$=Pr($\xi_{n+1} = 4) = 0$, P_{01}=Pr($\xi_{n+1} = 2) = 0.5$

P_{ij} =Pr($\xi_{n+1} = i - j$) , $s < i \le S$ for non-replenishment

$P_{1,-1}$=Pr($\xi_{n+1} = 2) = 0.5$, P_{11}=Pr($\xi_{n+1} = 0) = 0.3$, P_{21}=Pr($\xi_{n+1} = 1) = 0.2$

(b)

$$
\begin{array}{c} \\ (S,S) \\ (S,C) \\ (C,S) \\ (C,C) \end{array}
\begin{array}{cccc} (S,S) & (S,C) & (C,S) & (C,C) \\ \left\| \begin{array}{cccc} 0.7 & 0.3 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \end{array} \right\| \end{array}
$$

In the long run, the limiting distribution is $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$

$$0.7\pi_0 + 0.5\pi_2 = \pi_0 \Rightarrow \pi_2 = 0.6\pi_0 \qquad (1)$$
$$0.3\pi_0 + 0.5\pi_2 = \pi_1 \Rightarrow \pi_1 = 0.6\pi_0 \qquad (2)$$
$$0.6\pi_1 + 0.8\pi_3 = \pi_3 \Rightarrow \pi_3 = 1.8\pi_0 \qquad (3)$$

And $\because$ $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ (4)

$$\therefore \ \pi_0 = \frac{1}{4} = 0.25$$
$$\Rightarrow \pi = (0.25, 0.15, 0.15, 0.45)$$

The long run fraction of days in which it is cloudy is

$$\pi_2 + \pi_3 = 0.15 + 0.45$$
$$= 0.60$$

Q4: [4+4]

(a)

(i)

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{ccc} 0 & 1 & 2 \\ \left\| \begin{array}{ccc} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.5 \end{array} \right\| \end{array}
$$

$$\because \Pr\{X_2=1\} = \Pr\{X_2=1 | X_0=1\}\Pr\{X_0=1\}$$
$$= P_{11}^2 p_1$$
$$= P_{11}^2,\ p_1 = \Pr\{X_0=1\}=1$$

$$P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.49 & 0.28 & 0.23 \\ 0.43 & 0.22 & 0.35 \\ 0.47 & 0.20 & 0.33 \end{array} \right\| \end{array}$$

$$\therefore \Pr\{X_2=1\} = P_{11}^2$$
$$= 0.22$$

(ii) To get the limiting distribution $\pi = (\pi_0, \pi_1, \pi_2) = (\pi_G, \pi_D, \pi_R)$

$$\because \pi_j = \sum_{k=0}^{2} \pi_k P_{kj},\ j = 0,1,2$$

and $\sum_{k=0}^{2} \pi_k = 1$

$\Rightarrow$

$$\pi_0 = 0.6\pi_0 + 0.3\pi_1 + 0.4\pi_2$$

$$\pi_1 = 0.3\pi_0 + 0.3\pi_1 + 0.1\pi_2$$

$$\pi_2 = 0.1\pi_0 + 0.4\pi_1 + 0.5\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

Solving the following equations

$$4\pi_0 - 3\pi_1 - 4\pi_2 = 0 \qquad (1)$$
$$3\pi_0 - 7\pi_1 + \pi_2 = 0 \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (3)$$

By solving equations using Cramer's rule, we get

$$\Delta = \begin{vmatrix} 4 & -3 & -4 \\ 3 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -66,\ \Delta_0 = \begin{vmatrix} 0 & -3 & -4 \\ 0 & -7 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -31$$

$$\Delta_1 = \begin{vmatrix} 4 & 0 & -4 \\ 3 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -16,\ \Delta_2 = \begin{vmatrix} 4 & -3 & 0 \\ 3 & -7 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -19$$

$$\therefore\ \pi_0 = \tfrac{\Delta_0}{\Delta} = \tfrac{31}{66},\ \pi_1 = \tfrac{\Delta_1}{\Delta} = \tfrac{16}{66},\ \pi_2 = \tfrac{\Delta_2}{\Delta} = \tfrac{19}{66}$$

$\therefore$ The limitting distribution is $\pi = (\pi_0, \pi_1, \pi_2) = (31/66, 16/66, 19/66)$

(iii) $\pi_R = \pi_2 = \tfrac{19}{66} \simeq 0.2879$

(b)

For pure death process, the transition probabilities are given by

$$p_N(t) = e^{-\mu_N t} \qquad (1)$$

and for $n < N$

$$p_n(t) = pr\left\{X(t) = n \middle| X(0) = N\right\}$$
$$= \mu_{n+1}\mu_{n+2}\ \dots\ \mu_N\left[A_{n,n}e^{-\mu_n t} +\ \dots + A_{k,n}e^{-\mu_k t} +\ \dots + A_{N,n}e^{-\mu_N t}\right] \quad (2)$$

Where $A_{k,n} = \prod_{i=N}^{n} \dfrac{1}{(\mu_i - \mu_k)}$, $\ i \neq k,\ n \le k \le N,\ i = N, N\text{-}1,\ \dots,\ n \quad (3)$

For $N = 3 \quad (1) \Rightarrow p_3(t) = e^{-\mu_3 t}$

$\therefore\ p_3(t) = e^{-5t} \qquad$ (I)

For $n = 2 \quad (2) \Rightarrow p_2(t) = \mu_3\left[A_{2,2}e^{-\mu_2 t} + A_{3,2}e^{-\mu_3 t}\right]$

$$(3) \Rightarrow\ A_{2,2} = \prod_{i=3}^{2} \frac{1}{(\mu_i - \mu_2)}\ ,\ i \neq 2$$
$$= \frac{1}{\mu_3 - \mu_2} = \frac{1}{2}$$

$$,\ A_{3,2} = \prod_{i=3}^{2} \frac{1}{(\mu_i - \mu_3)},\ i \neq 3$$
$$= \frac{1}{\mu_2 - \mu_3} = -\frac{1}{2}$$

$\therefore$ $p_2(t)=5\left[\frac{1}{2}e^{-3t}-\frac{1}{2}e^{-5t}\right]$ (II)

For $n=1$ $(2)\Rightarrow p_1(t)=\mu_2\mu_3\left[A_{1,1}e^{-\mu_1 t}+A_{2,1}e^{-\mu_2 t}+A_{3,1}e^{-\mu_3 t}\right]$

$$(3)\Rightarrow A_{1,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_1)}\ ,\ i\neq 1$$

$$=\frac{1}{(\mu_3-\mu_1)(\mu_2-\mu_1)}=\frac{1}{3}$$

$$A_{2,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_2)}\ ,\ i\neq 2$$

$$=\frac{1}{(\mu_3-\mu_2)(\mu_1-\mu_2)}=-\frac{1}{2}$$

$$,\ A_{3,1}=\prod_{i=3}^{1}\frac{1}{(\mu_i-\mu_3)}\ ,\ i\neq 3$$

$$=\frac{1}{(\mu_2-\mu_3)(\mu_1-\mu_3)}=\frac{1}{6}$$

$\therefore$ $p_1(t)=15\left[\frac{1}{3}e^{-2t}-\frac{1}{2}e^{-3t}+\frac{1}{6}e^{-5t}\right]$ (III)

Using (I), (II) and (III) we can get $p_0(t)$ as follows

$$\therefore\ p_0(t)=1-\left[p_1(t)+p_2(t)+p_3(t)\right]$$

$$=1-\left[5e^{-2t}-\frac{15}{2}e^{-3t}+\frac{5}{2}e^{-3t}+\frac{5}{2}e^{-5t}-\frac{5}{2}e^{-5t}+e^{-5t}\right]$$

$$=1-5e^{-2t}+5e^{-3t}-e^{-5t} \qquad \text{(IV)}$$

Q5: [5+2]

(a)

$$\frac{dp_0(t)}{dt}=-\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt}=\lambda_{n-1}p_{n-1}(t)-\lambda_n p_n(t),\ n=1,2,3,\ldots \qquad (2)$$

The initial condition is $X(0)=1 \quad \Rightarrow \; p_1(0)=1$

$$\Rightarrow \; p_n(0)=\begin{cases} 1 & , \; n=1 \\ 0 & , \; \text{otherwise} \end{cases}$$

For $\lambda_0 = 0 \qquad (1) \Rightarrow \dfrac{dp_0(t)}{dt}=0$

$\therefore \; p_0(t)=0$, where $p_0(0)=0 \qquad\qquad (3)$

$$(2) \Rightarrow \; \frac{dp_n(t)}{dt}=\lambda_{n-1}p_{n-1}(t)-\lambda_n p_n(t)$$

$$\Rightarrow \; \frac{dp_n(t)}{dt}+\lambda_n p_n(t)=\lambda_{n-1}p_{n-1}(t), \; n=1,2, \; ...$$

$\because \; \lambda_n = n\lambda, \;\; \lambda_{n-1}=(n-1)\lambda$

$\therefore \; \dfrac{dp_n(t)}{dt}+n\lambda p_n(t)=(n-1)\lambda p_{n-1}(t),$ n=1,2, ...

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt}+n\lambda p_n(t)\right]=(n-1)\lambda p_{n-1}(t)e^{n\lambda t}$$

$$\therefore \; \frac{d}{dt}\left[p_n(t)e^{n\lambda t}\right]=(n-1)\lambda p_{n-1}(t)e^{n\lambda t}$$

$$\Rightarrow \int_0^t d\left[p_n(x)e^{n\lambda x}\right]=(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx$$

$$\therefore \; \left[p_n(x)e^{n\lambda x}\right]_0^t=(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx$$

$$\Rightarrow \; p_n(t)=e^{-n\lambda t}\left[p_n(0)+(n-1)\lambda\int_0^t p_{n-1}(x)e^{n\lambda x}dx\right], \; n=1,2, \; ... \quad (4)$$

which is a recurrence relation.

at $n=1$

$$p_1(t)=e^{-\lambda t}\left[p_1(0)+0\right]=e^{-\lambda t} \qquad (5)$$

at $n=2$

$$p_2(t)=e^{-2\lambda t}\left[p_2(0)+\lambda\int_0^t p_1(x)e^{2\lambda x}dx\right]$$

$(5) \Rightarrow \; p_1(x) = e^{-\lambda x}$

$\therefore \; p_2(t) = e^{-2\lambda t}\left[\lambda \int_0^t e^{-\lambda x} e^{2\lambda x} dx\right]$

$\therefore \; p_2(t) = \lambda e^{-2\lambda t} \int_0^t e^{\lambda x} dx$

$= e^{-\lambda t}(1 - e^{-\lambda t})^1 \qquad (6)$

Similarly as (5) and (6), we deduce that

$p_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$

$= p(1\text{-}p)^{n-1}, \;\; p = e^{-\lambda t}, \; n = 1, 2, \; ...$

$\therefore X(t) \sim geom(p), \; p = e^{-\lambda t}$

$Mean\left[X(t)\right] = 1/p = e^{\lambda t},$

$Variance\left[X(t)\right] = \dfrac{1-p}{p^2} = \dfrac{1-e^{-\lambda t}}{e^{-2\lambda t}}$

(b)

For Yule process,

$p_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1}, \quad n \ge 1$

$\Rightarrow$

$\because pr\left\{X(U) = k\right\} = \int_0^1 e^{-\beta u}(1 - e^{-\beta u})^{k-1} du$

$= \dfrac{1}{\beta}\int_0^1 (1 - e^{-\beta u})^{k-1}. \; \beta e^{-\beta u} du$

$= \dfrac{1}{\beta}\left[\dfrac{(1 - e^{-\beta u})^k}{k}\right]_0^1$

$= \dfrac{1}{\beta k}\left[(1 - e^{-\beta})^k\right]$

$\therefore pr\left\{X(U) = k\right\} = \dfrac{p^k}{\beta k}, \; k = 1, 2, \; ...$ where $p = 1 - e^{-\beta}$
