King Saud University
College of Science
Department of Mathematics
2^{nd} Semester 1446 H. (2025)

Math 380
Stochastic Processes
Second Midterm Exam
Time: 2 hours

Problem 1 [4 marks]:
Consider the Markov chain $X_0, X_1, \ . \ . \ .$ with transition probability matrix given by

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0 & 0.3 & 0.7 \\ 0.4 & 0 & 0.6 \end{pmatrix} \end{array}$$

Answer the following questions independently:

1. If the initial distribution is given by $p_0 = \mathbb{P}\{X_0 = 0\} = 0.2,\ p_1 = \mathbb{P}\{X_0 = 1\} = 0.5$, and $p_2 = \mathbb{P}\{X_0 = 2\} = 0.3$, then determine $\mathbb{P}\{X_0 = 0, X_1 = 1, X_2 = 2\}$.

2. Determine the conditional probabilities

$$\mathbb{P}\{X_2 = 1, X_3 = 1 \big| X_1 = 0\} \ \text{ and } \ \mathbb{P}\{X_1 = 1, X_2 = 1 \big| X_0 = 0\}.$$

3. If it is known that the process starts in state $X_0 = 2$, then determine the probabilities $\mathbb{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$ and $\mathbb{P}\{X_2 = 1\}$.

Problem 2 [3 marks]:
Let X_n denote the quality of the n^{th} item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective". Suppose that X_n evolves as a Markov chain, where a good item is followed by another good item with probability 0.8 and is followed by a defective item with probability 0.2, while a defective item is followed by another defective item with probability 0.6 and is followed by a good item with probability 0.4.

1. Find the corresponding transition probability matrix.

2. If the first item is good, what is the probability that the first defective item to appear is the fifth one?

3. What is the probability that the 4^{th} item is defective given that the 1^{st} is defective?

Problem 3 [3 marks]:
Consider the Gambler's ruin model, where the probability of the thrower winning in the dice game called "craps" is $p = 0.494$. Suppose that Player I is the thrower and begins the game with 10\$, and his opponent, Player II, begins with 20\$. Assuming that the bet is 1\$ per round, what is the probability that player I goes bankrupt before Player II?

!!! Please turn the page over !!!

Problem 4 [5 marks]:
Consider the following matrix:

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0.2 & \beta & 0.1 & 0.1 \\ \gamma & 0.2 & 0.4 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

1. Find α, β, γ, such that $\mathbf{P}$ defines a transition probability matrix of some Markov chain.
2. Assuming the values of α, β, γ obtained in (1), sketch the transition diagram, and classify the states (as absorbing or transient).
3. Starting in state 1, determine the probability that the Markov chain ends in state 0, as well as the mean time to absorption.

Problem 5 [5 marks]:
An auto insurance company classifies its customers in three categories: poor, satisfactory and preferred. Assume the following happens within one year:

- No customer is upgraded from poor to preferred category, or is downgraded from preferred to poor category.
- 40% of poor category become satisfactory.
- 30% of satisfactory customers move to preferred category.
- 10% of satisfactory customers become poor.
- 20% of preferred customers are downgraded to satisfactory.

1. Write the transition probability matrix associated to this model.
2. What is the long run fraction of drivers in each of these three categories.

Problem 6 [5 marks]:
Consider the following stock market model, where the price of the stock is recorded at the end of a given day, assuming that the stock's value will be increased tomorrow depending upon whether it has increased today and yesterday. If the stock has increased for the last two days, then it will increase tomorrow with probability 0.8. If the stock has increased today but has decreased yesterday, then it will increase tomorrow with probability 0.6. If the stock has decreased today but has increased yesterday, then it will increase tomorrow with probability 0.4. If the stock has decreased for the last two days, then it will increase tomorrow with probability 0.2.

1. Transform this model into a Markov chain. Then, find its transition probability matrix, and sketch the corresponding transition diagram.
2. Find the long run fraction of days in which the value of the stock increases.
3. Find the long run fraction of days in which the value of the stock decreases.

Good Luck