King Saud University
College of Science
Department of Mathematics
2^{nd} Semester 1446 H. (2025)

Math 380
Stochastic Processes
First Midterm Exam
Time: 2 hours

Problem 1 [4 marks]:
An insurance company divides its policyholders into low-risk and high-risk classes. For the year, of those in the low-risk class, 75% had no claims, 17% had one claim, and 8% had two claims. Of those in the high-risk class, 53% had no claims, 25% had one claim, and 22% had two claims. Of the policyholders, 63% were in the low-risk class and 37% in the high-risk class.

1. If a policyholder had no claims in the year, what is the probability that he is in the low-risk class?
2. If a policyholder had two claims in the year, what is the probability that he is in the high-risk class?

Problem 2 [2 marks]:
The GPA random variable X assigns to the letter grades A, B, C, D and F the numerical values $4, 3, 2, 1$ and 0. Find the expected value of X for a student selected at random from a class in which there were 20 A grades, 36 B grades, 51 C grades, 10 D grades, and 3 F grades.

Problem 3 [4 marks]:

1. If X is a Poisson random variable with parameter $\lambda > 0$, then calculate the expectation $\mathbb{E}[X]$.
2. Policyholders of an insurance company file claims at an average rate of 0.4 per year. If the company pays 5000 \$ for each claim, what is the mean claim amount for a policyholder in a year?

Problem 4: [5 marks]
Consider the function

$$f(x) = \begin{cases} \alpha\,(0.3x + 0.05)\,, & \text{for } 0 \le x \le 1, \\ \\ 0, & \text{elsewhere.} \end{cases}$$

1. Find the value of the constant α so that $f(x)$ is a probability density function of some random variable X.
2. Find the associated cumulative distribution function.
3. Calculate $\mathbb{P}\left(0 \le X \le \frac{1}{2}\right)$ and $\mathbb{P}\left(\frac{1}{4} \le X \le \frac{3}{4}\right)$.
4. Given that X exceeds $\dfrac{1}{3}$, what is the probability that X exceeds $\dfrac{1}{2}$?
5. Calculate $\mathbb{E}[X]$ and $\mathbf{Var}[X]$.

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Problem 5 [5 marks]:

1. If the growth is assumed to be continuous at rate r, then, the value of an asset at time t is given by: $A(t) = A(0)\ e^{rt}$. Suppose that such stock was purchased for $A(0) = 100$, and the continuous growth rate is 20% per year.

   (a) Find the value of the stock in 5 months.

   (b) Find the value of the stock in one year.

2. For future values of the stock, the rate of growth is a random variable X. If we assume that this rate X is normally distributed, then the future value of the stock in one year is given by $Y = A(0)e^X$. Suppose that such stock was initially purchased for $A(0) = 100$, and its value grows according to a normal random rate X with parameters $\mu = 0.2$ and $\sigma = 0.04$, then:

   (a) What is the probability that the value of the stock in one year will be greater than 123.70?

   (b) What is the probability that the value of the stock in one year will be less than 130.50?

Problem 6 [5 marks]:

1. Let $X_1, X_2, \ldots,$ be independent random variables such that $X_j \geq 0$ and $\mathbb{E}[X_j] = 1$, for $j = 1, 2, \ldots$.
   Show that the stochastic process defined by

$$G_n := \begin{cases} G_0 > 0, & n = 0, \\ G_0 X_1 X_2 \ldots X_n, & n \geq 1. \end{cases}$$

   is a martingale.

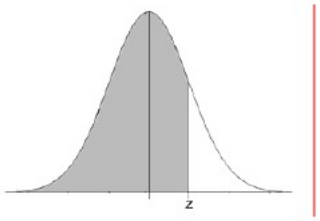
2. Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables. Let $m(t) = \mathbb{E}[e^{tX_1}] < \infty$ be the moment generating function of X_1. Either by deducing it from (1), or by proving it directly, show that the stochastic process

$$T_n := (m(t))^{-n}\, e^{t(X_1 + X_2 + \ldots + X_n)}, n \geq 0,$$

   defines a martingale (Justify your work rigorously).

Good Luck

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998