King Saud University
College of Science
Department of Mathematics
2^{nd} Semester 1446 H. (2025)

Math 380
Stochastic Processes
Final Exam
Time: 3 hours

Problem 1 [10 marks]:
Consider the following matrix:

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0.1 & \beta & 0.1 & 0.3 \\ 0.4 & 0.3 & \gamma & 0.1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \end{array}.$$

1. Find $\alpha,\ \beta,\ \gamma,\ \lambda$, such that $\mathbf{P}$ defines a transition probability matrix of some Markov chain $\{X_t\}$.
2. Assuming the values of $\alpha, \beta, \gamma, \lambda$ obtained in (1), sketch its transition diagram, and classify its states (as absorbing or transient).
3. Determine the conditional probability $\mathbb{P}\{X_1 = 2, X_2 = 2 \big| X_0 = 1\}$.
4. Starting in state 1, determine the probability that this Markov chain ends in state 3, as well as the mean time to absorption.

Problem 2 [9 marks]:
Consider the reliability model, where a hotel reception counter has two computers, only one of which is in operation at any given time. A computer may break down on any given day with probability 0.2. It takes 2 days to restore a computer to normal. Suppose that there is a duplicate repair facility so that both computers can be repaired at the same time. Then, a Markov chain can be formed by taking states as the pairs (i, j), where i is the number of machines in operating condition at the end of a day, and j is 1 if a day's labour has been expended on a machine not yet repaired, and 0 otherwise.

1. Find the transition probability matrix.
2. Find the probability in the long run that neither computer is operating.
3. What is the availability that at least one computer is operating in the long run?
4. Suppose that every period that this process spends in state $(2, 0)$ incurs a cost of \$4. Every period that the process spends in state $(1, 0)$ incurs a cost of \$14. Every period that the process spends in state $(1, 1)$ incurs a cost of \$24. Every period that the process spends in state $(0, 1)$ incurs a cost of \$30. What is the long run mean cost per period associated with this Markov chain?

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Problem 3 [9 marks]:

1. Let $\{X(t) : \ t \geq 0\}$ be a nonhomogeneous Poisson process with intensity function
$$\lambda(t) = \begin{cases} t, & \text{for } \ 0 < t < 20 \\ 20, & \text{for } \ t > 20. \end{cases}$$
Calculate the expected number of event occurrences during the period from time 10 until time 30.

2. Consider the Yule process with infinitesimal parameters $\lambda_n = 2n, \ n \geq 1,$ (here $\beta = 2$), and $X(0) = 1$.
    (a) Determine $P_n(t)$, for $n = 1, \ 2, \ 3,$ (you can use the formula without proof).
    (b) Calculate its mean and variance.

3. An aggregate loss distribution has compound Poisson distribution with expected number of claims equal to 2. Suppose that individual claim amounts can take only the values $1, 2, 3$ with equal probabilities. Determine the probability that aggregate losses exceed 3.

Problem 4 [12 marks]:

1. Let $B(t)$ denote the standard Brownian motion, and consider the stochastic process $X(t) = 2 + 3B(t), \ \ t \in [0, \infty)$. Find the probability density function of the process $X(t)$ at time t.

2. Let $B(t)$ be a Brownian motion. Show that the process $X(t) = -B(t)$ is also a Brownian motion on $[0, T]$.

3. Let $B(t)$ be the standard Brownian motion. Show that $B^3(t) - 3tB(t)$ is a martingale.

4. Let $B(t)$ be the standard Brownian motion. Show that the following related process $X(t) = e^{aB(t)+bt}, \ t \geq 0$, is a martingale if and only if $\frac{a^2}{2} + b = 0$.

Good Luck