King Saud University
College of Science
Department of Mathematics
1st Sem. 1447 H. (25/26)

Math 380
Stochastic Processes
Second Midterm Exam
Time: 2 hours

Problem 1 [6 marks]:
Consider the Markov chain $X_0, X_1, \ . \ . \ . \ ,$ with transition probability matrix given by

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{pmatrix} \end{array}$$

Answer the following questions <u>independently</u>:

1. If the initial distribution is $\mathbb{p}_0 = \mathbb{P}\{X_0 = 0\} = 0.5, \ \mathbb{p}_1 = \mathbb{P}\{X_0 = 1\} = 0.5$, and $\mathbb{p}_2 = \mathbb{P}\{X_0 = 2\} = 0$, then calculate the probabilities $\mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 0]$ and $\mathbb{P}\{X_0 = 0, X_1 = 1, X_2 = 2\}$, as well as $\mathbb{P}[X_0 = 1, X_1 = 1, X_3 = 0]$.

2. Determine the conditional probabilities
$$\mathbb{P}\{X_2 = 1, X_3 = 1 \big| X_1 = 0\} \ \text{ and } \ \mathbb{P}\{X_1 = 1, X_2 = 1 \big| X_0 = 0\}.$$

3. If it is known that the process starts in state $X_0 = 2$, then determine the probabilities $\mathbb{P}\{X_0 = 2, X_1 = 1, X_2 = 0\}$ and $\mathbb{P}\{X_2 = 1\}$.

Problem 2 [3 marks]:
Consider the daily rainfall model, assume that the probability of being rainy tomorrow is 0.75 if it is raining today, and assume that the probability of it will be rainy tomorrow is 0.45 if it is clear (dry) today. Also assume that these probabilities do not change if information is also provided about the weather before today.

1. Explain why the stated assumptions imply that the Markov property holds for this weather evolution model.

2. Formulate the evolution of the weather as a Markov chain X_n by defining its states and giving its one-step transition matrix.

3. Draw the state transition diagram for this Markov chain.

4. Calculate $\mathbb{P}[X_n = \text{rainy} | X_0 = \text{rainy}]$, for $n = 0, 1, 2$.

Problem 3 [3 marks]:
Consider the population extinction model, where we suppose that a parent has no offspring with probability $\frac{1}{3}$ and has two offspring with probability $\frac{2}{3}$. If a population of such individuals begins with 1 parent and evolves as branching process, determine the probability u_n that the population is extinct by the n^{th} generation for $n = 1, \ 2, \ 3, \ 4, \ 5$.

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Problem 4 [6 marks]:
Consider the following transition probability matrix :

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0.1 & 0.2 & \beta & 0.2 \\ 0.1 & 0.2 & 0.6 & \gamma \\ 0.2 & \lambda & 0.3 & 0.3 \end{pmatrix} \end{array}$$

1. Find the values of the constants $\alpha,\ \beta,\ \gamma,\ \lambda$, such that $\mathbf{P}$ defines a transition probability matrix of some Markov chain.
2. Assuming the values of $\alpha, \beta, \gamma, \lambda$ obtained in (1), sketch the transition diagram, and classify the states (as absorbing or communicating).
3. Starting in state $X_0 = 2$, determine the probability that the Markov chain ends in state 0, as well as the mean time to absorption.
4. Starting in state $X_0 = 1$, determine the probability that the process never visits the state 2. Justify your answer.

Problem 5 [2 marks]:
Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{pmatrix} 0.2 & 0.1 & 0.2 & 0.2 & 0.1 & 0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Compute, in the long run, the probability of being in state 0.

Problem 6 [5 marks]:
Consider the following stock market model, where the price of the stock is recorded at the end of a given day, assuming that the stock's value will be increased tomorrow depending upon whether it has increased today and yesterday. If the stock has increased for the last two days, then it will increase tomorrow with probability 0.75. If the stock has increased today but has decreased yesterday, then it will increase tomorrow with probability 0.65. If the stock has decreased today but has increased yesterday, then it will increase tomorrow with probability 0.35. If the stock has decreased for the last two days, then it will increase tomorrow with probability 0.25.

1. Transform this model into a Markov chain. Then, find its transition probability matrix, and sketch the corresponding transition diagram.
2. Find the long run fraction of days in which the value of the stock increases.
3. Find the long run fraction of days in which the value of the stock decreases.

Good Luck