King Saud University
College of Science
Department of Mathematics
1^{st} Sem. 1447 H. (25/26)

Math 380
Stochastic Processes
First Midterm Exam
Time: 2 hours

Problem 1 [4 marks]:
15% of a company's life insurance policyholders are smokers, while the rest are nonsmokers. For each smoker, the probability of dying during the year is 0.08. For each nonsmoker, the probability of dying during the year is 0.02.

1. Given that a policyholder has died during the year, what is the probability that this policyholder was a smoker?
2. Given that a policyholder has not died during the year, what is the probability that this policyholder was a nonsmoker?

Problem 2 [3 marks]:
An actuary has discovered that policyholders are five times as likely to file three claims as to file five claims. If the number of claims filed has a Poisson distribution, what is the variance of the number of claims filed?

Problem 3 [5 marks]:

1. State the memoryless property for a random variable X.
2. Show that an exponential random variable $X \sim \exp(\lambda)$ is memoryless.
3. Give a necessary and sufficient condition for a continuous random variable to satisfy the memoryless property.
4. A piece of equipment is being insured against early failure. The time from purchase until failure of the equipment is exponentially distributed with mean 10 years. The insurance will pay an amount x if the equipment fails during the first year, and it will pay $\frac{1}{2}x$ if failure occurs during the second or third year. If failure occurs after the first three years, no reimbursement will be made.
   (a) At what level must x be set if the expected payment made under this insurance is to be 1000?
   (b) If supposed to fail after 15 years, what is the probability that the equipment will fail by 25 years?

Problem 4: [3 marks]
Suppose that the claim severity random variable for an insurance company follows a lognormal distribution, and the normally distributed exponent has mean 7.75 and standard deviation 0.7. What is the probability that a claim is greater than \$1860 ?

!!! Please turn the page over !!!

Problem 5: [6 marks]
Let X and Y be two random variables with joint density function:

$$f_{XY}(x,y) = \begin{cases} \alpha(1-y), & \text{for } 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the value of the constant α so that f_{XY} defines a joint density function.
2. Calculate the joint probability $\mathbb{P}\left(X \le \frac{3}{4}\ ,\ Y \ge \frac{1}{2}\right)$.
3. Find the marginal densities $f_X(x)$ and $f_Y(y)$.
4. Are X and Y independent?
5. Calculate the conditional expectation $\mathbb{E}[X|Y=y]$.

Problem 6 [4 marks]:

1. Let $X_1, X_2, \ . \ . \ . \ ,$ be independent and identically distributed (i.i.d.) random variables with $\mathbb{E}[X_i] = 0$ and $\mathbf{Var}[X_i] = \mathbb{E}[X_i^2] = \sigma^2$. Consider their partial sum S_n defined as follows:

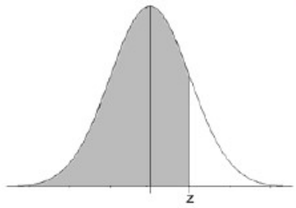
   $$\begin{cases} S_0 = 0, \\ S_n = X_1 + \ . \ . \ . \ + X_n, \ n \ge 1. \end{cases}$$

   Show that $M_n = S_n^2 - n\sigma^2$ is a martingale, (Justify your work rigorously).

2. Let $X_0 = 0, X_n = \varepsilon_1 + \varepsilon_2 + \ . \ . \ . \ \varepsilon_n, \ n = 1, 2, \ . \ . \ . \ ,$ where (ε_j) is an i.i.d. collection of Bernoulli random variables $\in \{\pm 1\}$ with $p = \frac{1}{2}$. Determine for which values of $\alpha \ge 0$ the stochastic process $Y_n := e^{X_n - \alpha n}$ defines a martingale.

Good Luck

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998